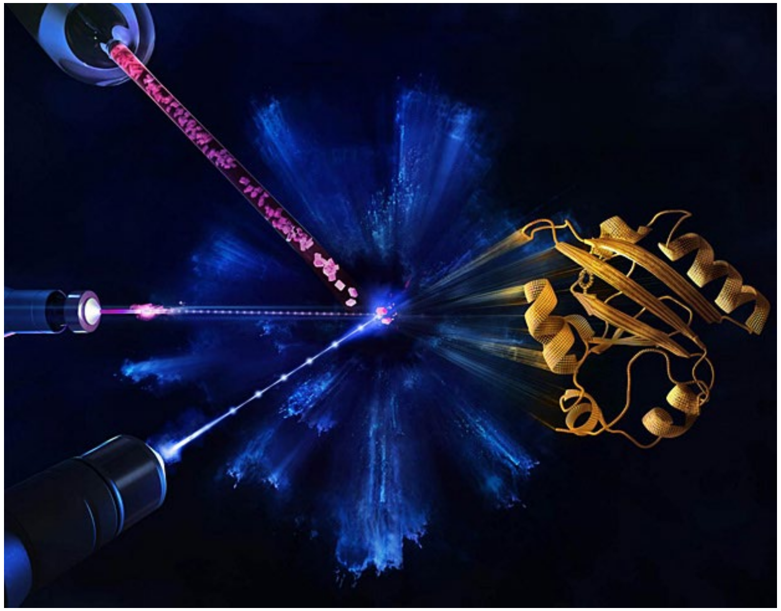
Fundamentals of Free-electron Laser



*Photograph adapted from

X-ray Free-Electron Laser Imaging Of Enzymes
•Press Release - Source: University of Wisconsin-Milwaukee, Posted November 18, 2019 1:07 PM

http://astrobiology.com/2019/11/x-ray-free-electron-laser-imaging-of-enzymes.html

"microcrystals are injected (top, left) and a reaction is initiated by blue laser pulses hitting the proteins within the crystals (middle, left). The atomic structure of the protein (right) is probed during the reaction by the X-ray pulses (bottom, left)."

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Outlines

Part I – fast-wave (Undulator) FEL

1. Spontaneous emission – Compton scattering/Thompson scattering/undulator radiation

2. Stimulated emission – wave/particle energy exchange $\rightarrow$ laser gain

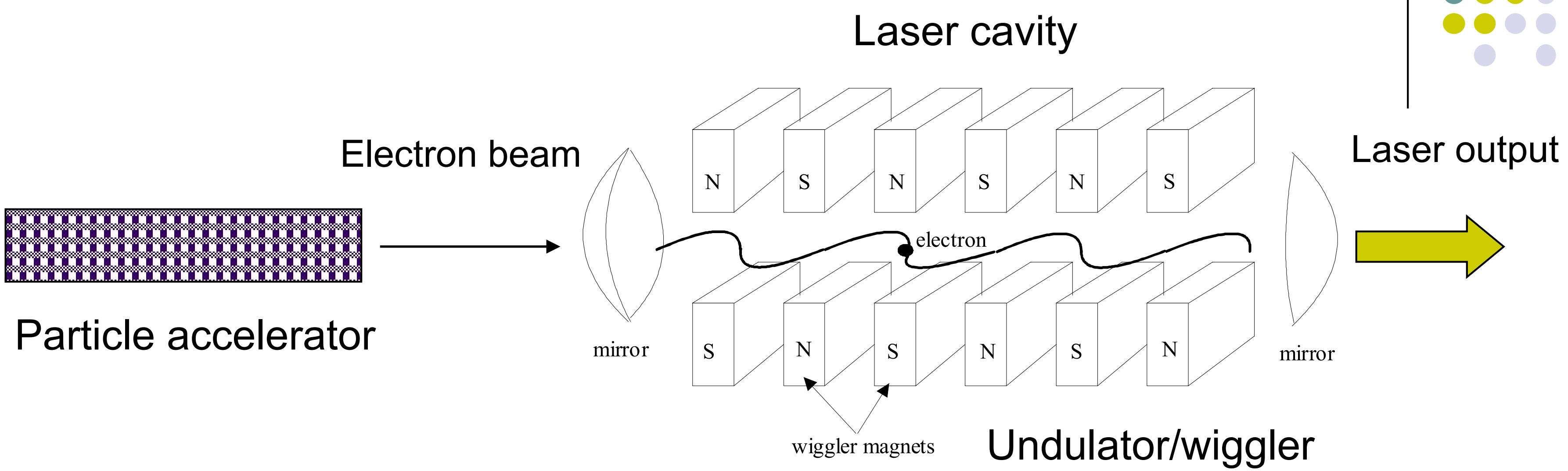
3. Requirements for FEL Oscillator: buildup time, energy spread, emittance, saturation power, etc.

Part II – Slow-wave FEL

Cherenkov, Traveling-wave Tube (TWT), Backward-wave Oscillator (BWO), Smith-Purcell FEL

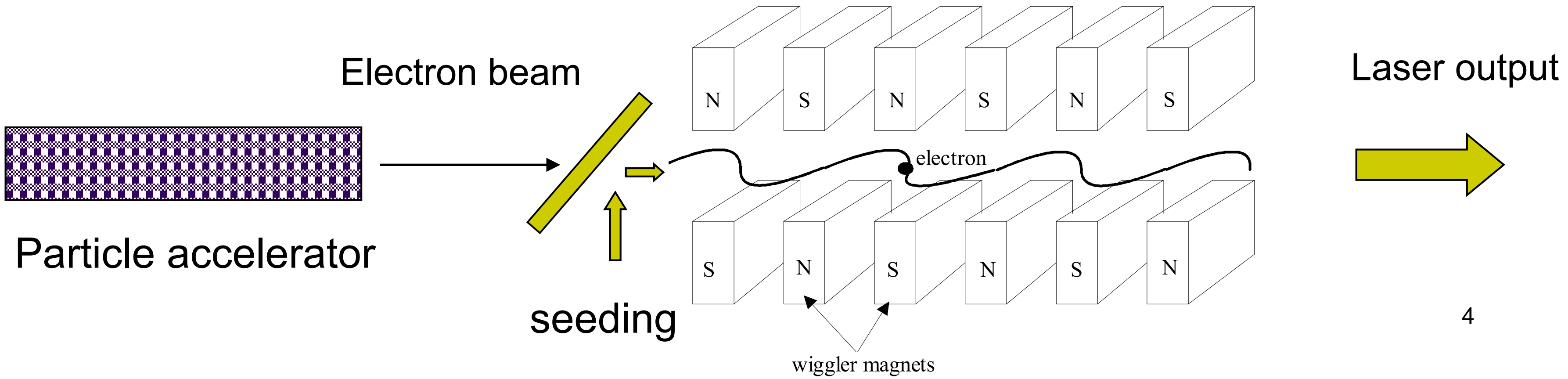
Part I – Fast-wave (Undulator) FEL

Low gain → free-electron Laser Oscillator

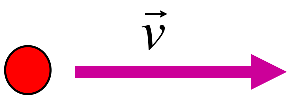


High gain → free-electron Laser Amplifier

Self-amplified Spontaneous emission (SASE) FEL



Parameters in Relativistic Mechanics



Moving particle

Lorentz factor $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$

where $\beta \equiv v/c$, with c = speed of light in vacuum.

Electron mass in motion: $m = \gamma m_0$, m_0 = electron rest mass 9.1×10^{-31} kg

Electron momentum: $p = mv = \gamma m_0 v$

Total electron energy: $\gamma m_0 c^2 = \sqrt{m_0^2 c^4 + p^2 c^2}$, $m_0 c^2$ = electron rest energy ~ 0.51 MeV

In laboratory frame: length L

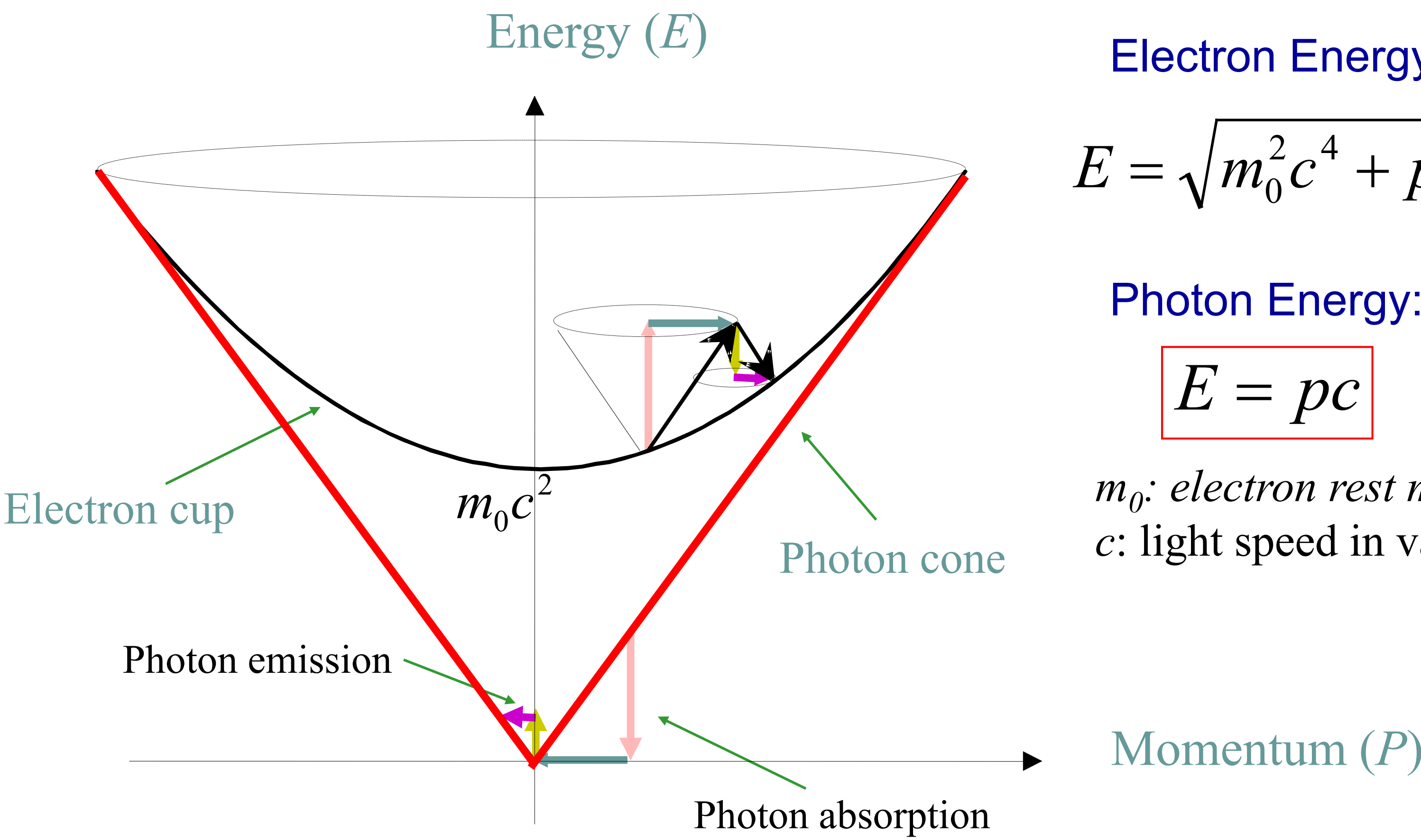
In electron frame: length $L/\gamma \leftarrow$ Lorentz contraction

In the relativistic regime $\beta \equiv v/c \sim< 1$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} >> 1 \Rightarrow \frac{1}{\beta} \sim 1 + \frac{1}{2\gamma^2} \Rightarrow \beta \sim 1 - \frac{1}{2\gamma^2}$$

Photon-electron Energy Exchange in Free Space

requirements: energy conversation & momentum conservation



Electron Energy:

$$E = \sqrt{m_0^2 c^4 + p^2 c^2}$$

Photon Energy:

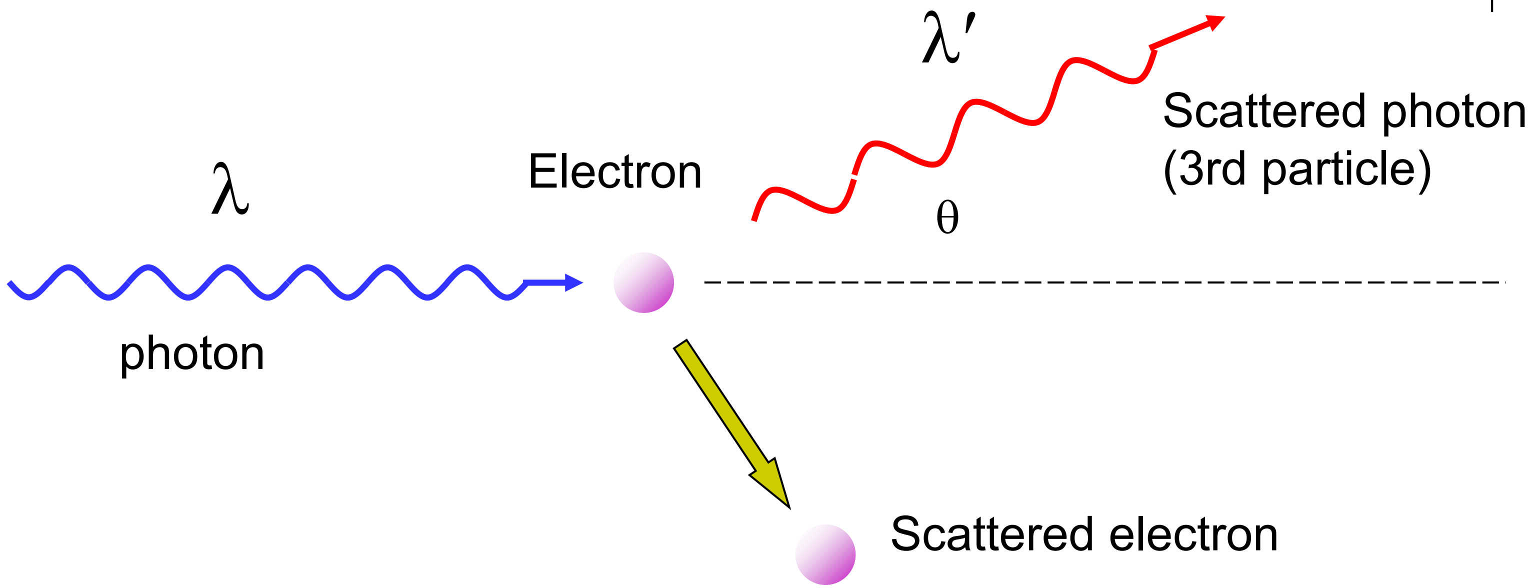
$$\boxed{E = pc}$$

m_0: *electron rest mass*
c: light speed in vacuum

Energy-momentum diagram of Compton Scattering

Photon-electron energy exchange is prohibited in a vacuum unless a third particle exists or is created

Compton Scattering

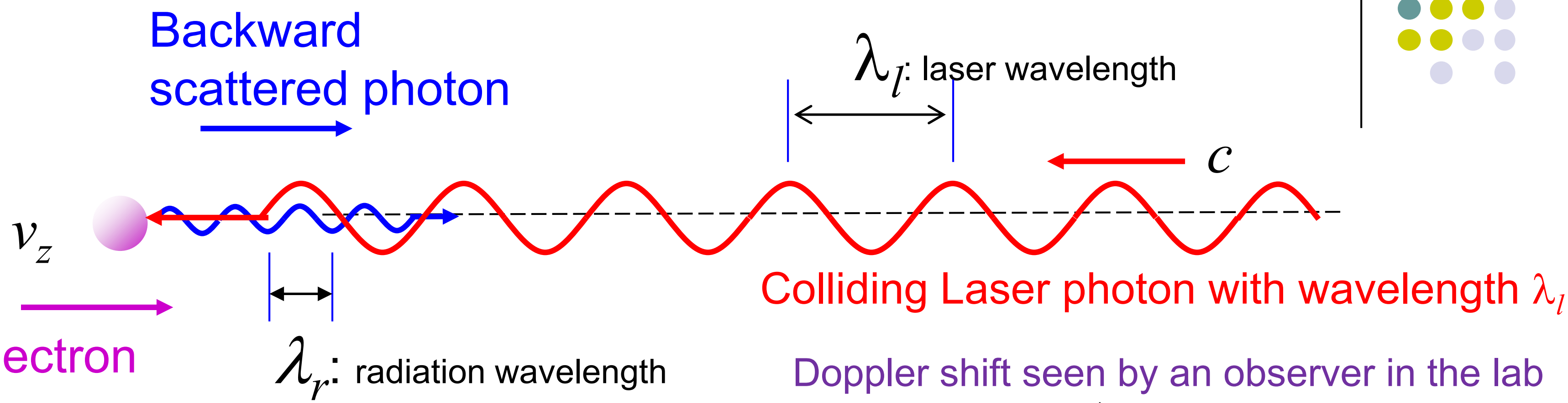


λ: wavelength
h: Planck's constant
m_0: electron rest mass
c: vacuum wave speed
p: momentum

Compton Effect $$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

Thomson Back Scattering:

Compton scattering with electron energy loss much less than the photon energy



Double Doppler shift
$$\underbrace{f_r}_{\text{Radiation frequency in the lab}} = \underbrace{f_l\sqrt{\frac{1+\beta_z}{1-\beta_z}}}_{f_e'\text{: Doppler shifted laser frequency seen by electron}} \cdot \sqrt{\frac{1+\beta_z}{1-\beta_z}} = f_e' \times \overbrace{\sqrt{\frac{1+\beta_z}{1-\beta_z}}}^{\text{Doppler shift seen by an observer in the lab}} \Rightarrow \lambda_r = \frac{\lambda_l}{4\gamma_z^2}$$

Radiation frequency in the lab

f_e' : Doppler shifted laser frequency seen by electron

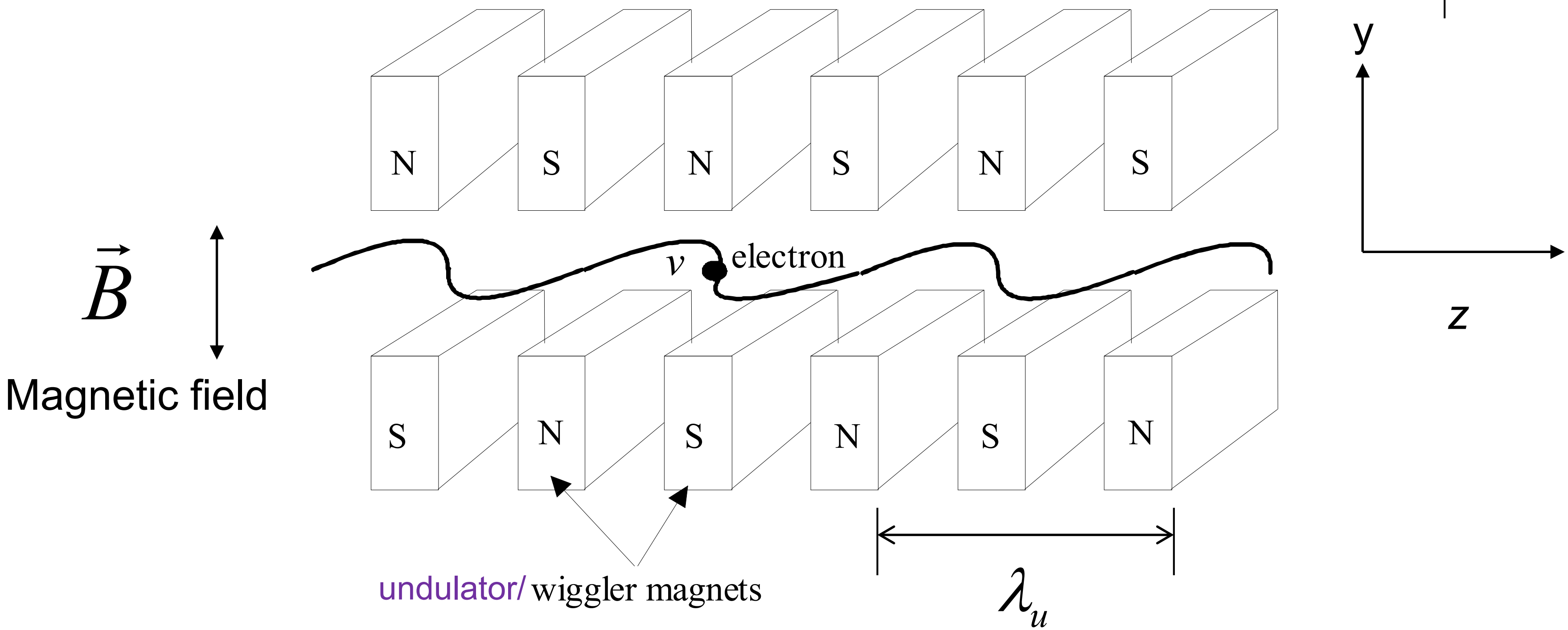
Given $\lambda = 800$ nm (Ti:sapphire laser), $\gamma \sim \gamma_z = 45$ (23 MeV beam), $\lambda_r = 1$ Å (hard x-ray!)

Longitudinal Lorentz factor $\gamma_z \equiv \dfrac{1}{\sqrt{1-\beta_z^2}}$

where $\beta_z \equiv v_z / c$

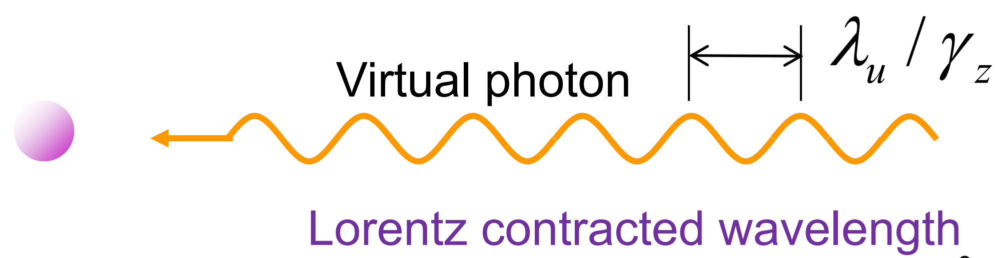
Undulator Radiation

In laboratory frame



In the electron rest frame, the electron sees a "wave" with fields:

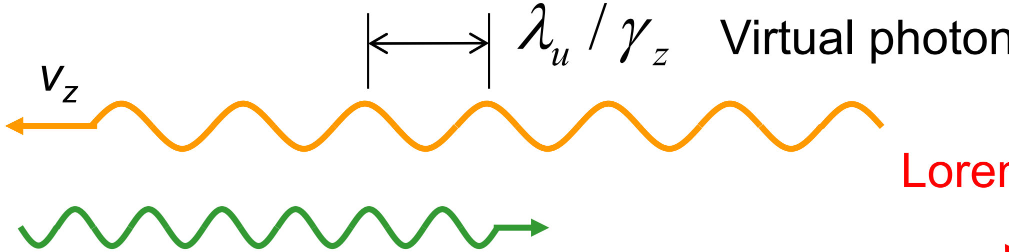
$$\vec{E}' = \gamma\vec{\beta}\times\vec{B}, \vec{B}' = \gamma\vec{B}$$



Spontaneous Undulator Radiation

Electron Rest Frame

electron

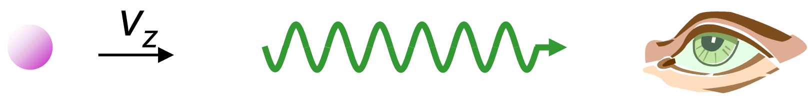


Electron oscillation frequency in the electron frame

$$f' = (T')^{-1} = \left(\frac{\lambda_u \boxed{/\gamma_z}}{v_z} \right)^{-1}$$

Lorentz contraction

Laboratory Frame



observer

Doppler Shift $$f = f' \sqrt{\frac{1+\beta_z}{1-\beta_z}}$$

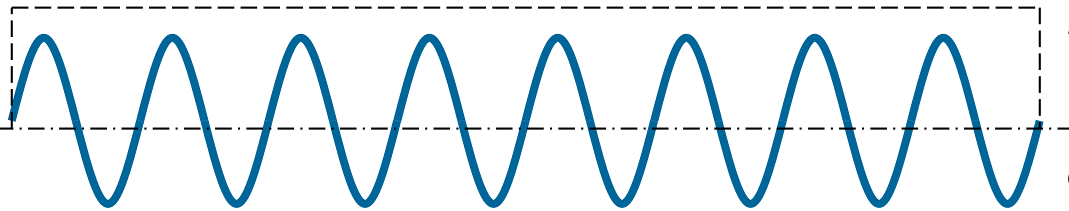
$$\boxed{\lambda = \lambda_u \left(\frac{1}{\beta_z} - 1 \right) \approx \frac{\lambda_u}{2\gamma_z^2}}$$

For $\lambda_u \sim 1$ cm, 100 MeV($\gamma_z \sim 200$), $\Rightarrow \lambda = 125$ nm

"Cheap" long-wavelength virtual photon $\Rightarrow$ expensive short-wavelength photon

Spontaneous Undulator Radiation

$$\tau = \frac{L_r}{c},\ L_r = N_u \lambda_r \qquad e^{i\omega_r t} \times \mathrm{rect}[\frac{t}{\tau}]$$



N_u: *number of undulator periods*
L_r: spatial radiation length (slippage distance)
τ: temporal radiation length
ω_r: resonant radiation frequency

Radiation Spectrum – Fourier transform of the temporal pulse

Fourier transform of a rectangular function

Rectangular function: $\mathrm{rect}[t] = \begin{cases} 1 & \text{for } |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$

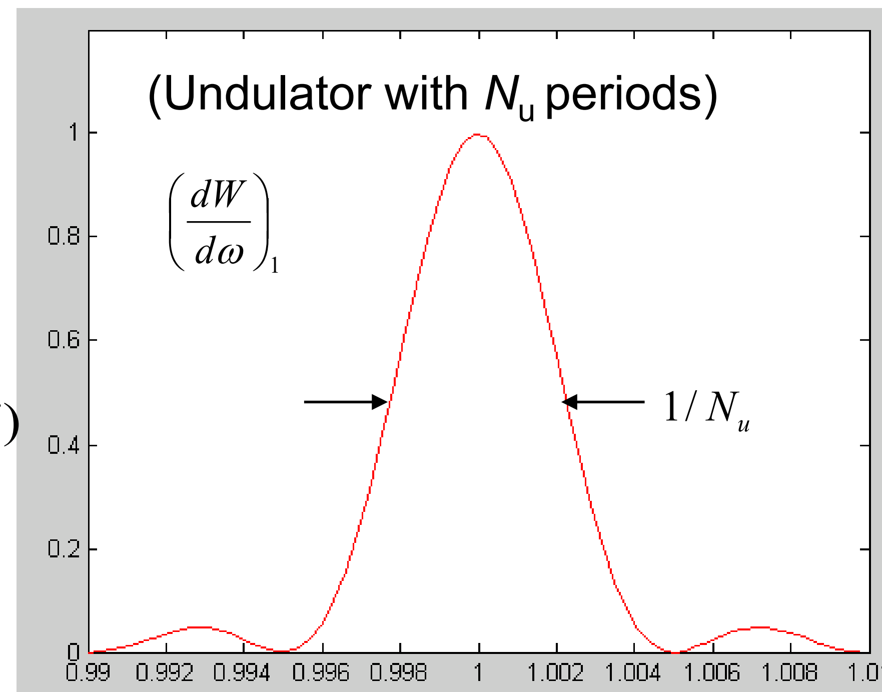
$$\text{Fourier Transform}\{\mathrm{rect}[t]\} = \frac{\sin(\omega/2)}{\omega/2} = \mathrm{sinc}(f)$$

Fourier transform of a ω_r modulated square function

$$\text{Fourier Transform} \quad \{e^{i\omega_r t} \times \mathrm{rect}[\frac{t}{\tau}]\}$$

$$\propto \frac{\sin[2N_u(\omega/\omega_r - 1)]}{2N_u(\omega/\omega_r - 1)}$$

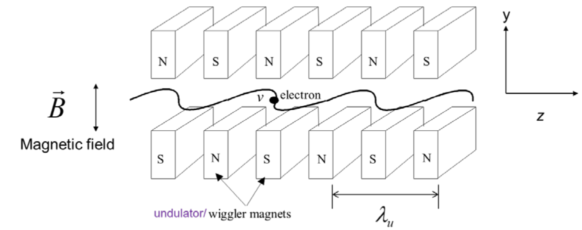
Spectral-energy Lineshape Function



ω/ω_r

$$\text{Spectral energy} \quad \left(\frac{dW}{d\omega}\right)_1 \propto \{\mathrm{sinc}[2N_u(\omega/\omega_r - 1)/\pi]\}^2$$

Effect of Magnetic field on e⁻ Quiver Motion



A general assumption: a relativistic beam $\gamma >> 1$

Assume a planar/linear wiggler with a wiggler field of $\vec{B} = \hat{y}\sqrt{2}B_{rms}\sin k_u z$

Begin with the Lorentz force equation $\dfrac{d\vec{p}}{dt} = e\vec{v}\times\vec{B}$, where $\vec{p} = \gamma m_0 \vec{v}$

$$v^2 = v_x^2 + v_x^2 .$$

$$v_x \approx \frac{-\sqrt{2}ca_u}{\gamma}\cos(k_u z) = \frac{-\sqrt{2}ca_u}{\gamma}\cos(k_u v_z t)$$

Wiggler wavenumber

x oscillation frequency

$$\boxed{k_u = 2\pi/\lambda_u}$$

$$v_z = \sqrt{v^2 - v_x^2} \approx \overset{\sim c}{v} - \frac{a_u^2}{2\gamma^2}\frac{c}{\beta}\cos(2k_u z)$$

$$= v - \frac{a_u^2}{2\gamma^2}\frac{c}{\beta}\cos(2k_u v_z t)$$

$\surd$

z oscillation frequency

where $a_u = \dfrac{eB_{rms}}{m_0 c k_u}, \varphi \sim \left(\dfrac{v_x}{v_z(\sim c)}\right)_{rms} = \dfrac{a_u}{\gamma}$

Wiggler parameter propagation angle

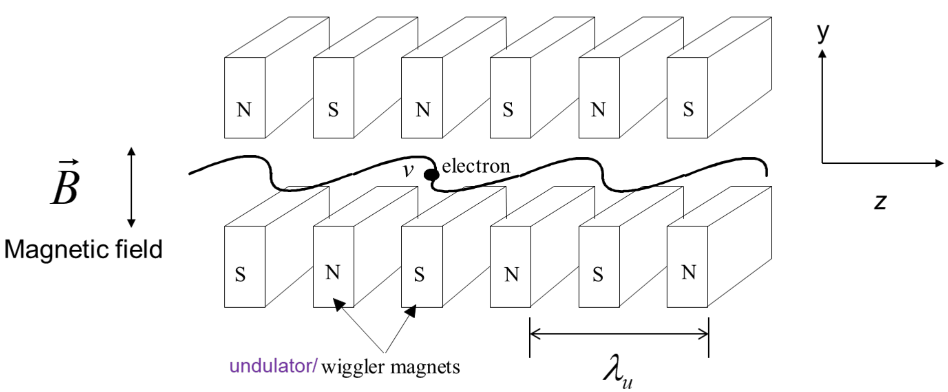
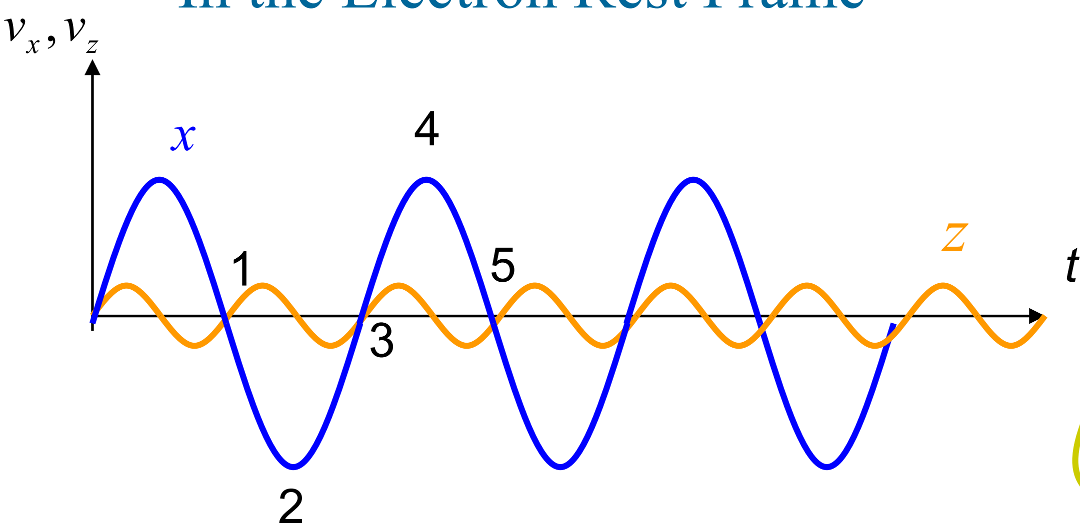
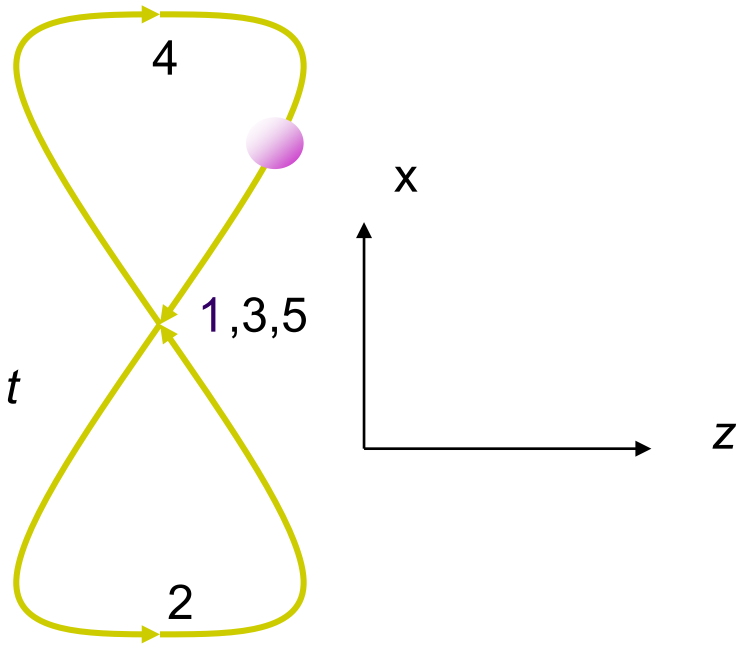


figure-8 motion

In the Electron Rest Frame



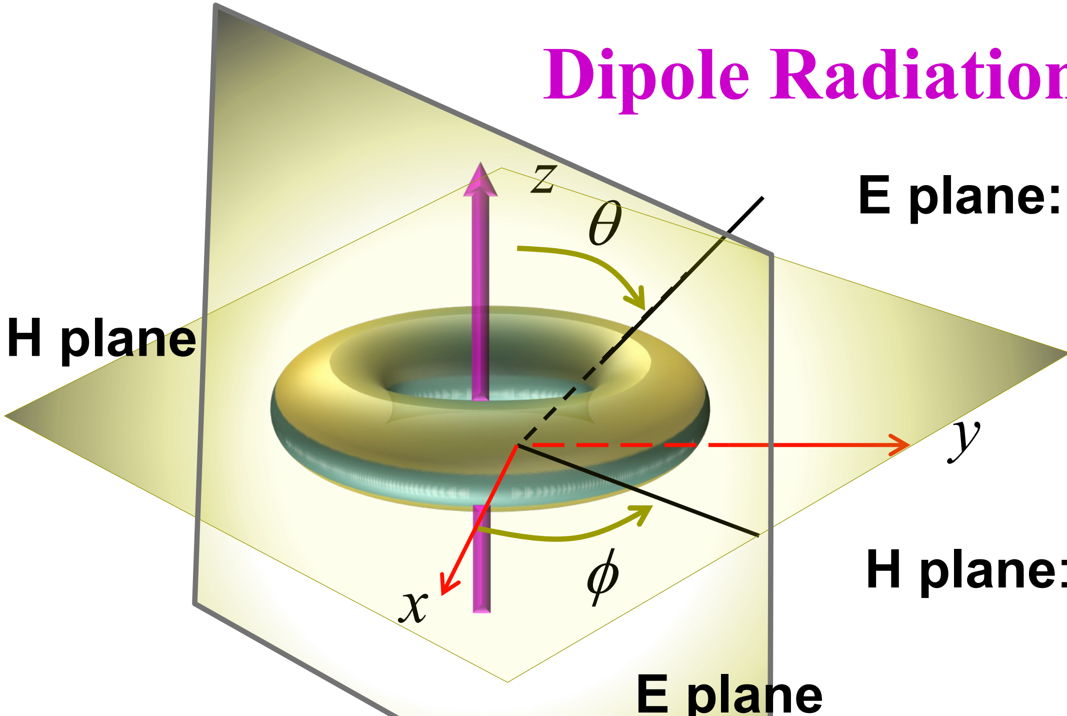


$\vec{v}_{z,2\omega} \times \vec{B}_{y,\omega}$ generates $\vec{v}_{x,3\omega}$, $\vec{v}_{x,3\omega} \times \vec{B}_{y,\omega}$ generates $\vec{v}_{z,4\omega}$ ……..

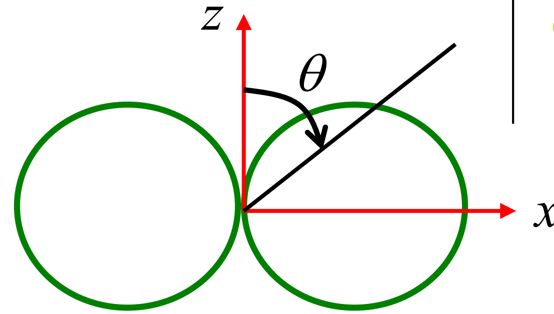
x oscillation frequencies: ω, 3ω, 5ω, 7ω, 9ω …..

z oscillation frequencies: 2ω, 4ω, 6ω, 8ω, 6ω …..

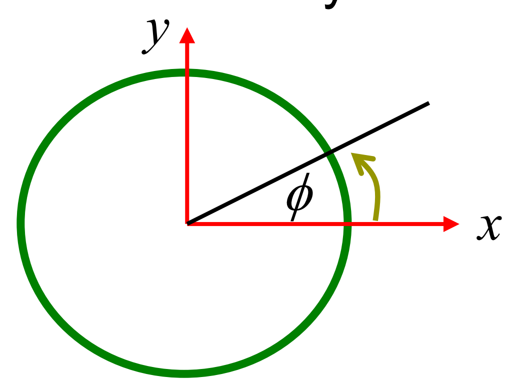
Dipole Radiation Pattern



E plane: the plane defined for a constant ϕ



H plane: the plane defined by $\theta = \pi/2$



Radiation intensity

$$S_{R,av}(R,\theta,\phi) \propto \sin^2\theta$$

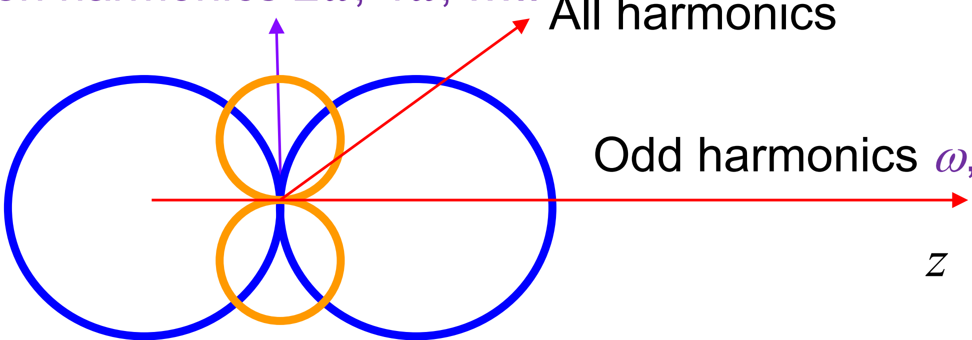
Dipole radiation pattern in the electron rest frame

Dipole radiation pattern in the laboratory frame

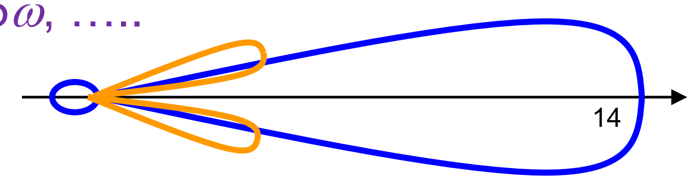
Even harmonics 2ω, 4ω, …..

All harmonics

Odd harmonics ω, 3ω, 5ω, …..



($\gamma >> 1$)



Undulator Radiation Wavelength

Because

$$\gamma_z \equiv \frac{1}{\sqrt{1-\beta_z^2}} = \frac{1}{\sqrt{1-v_z^2/c^2}}, \text{ and } \quad \lambda \approx \frac{\lambda_u}{2\gamma_z^2}$$

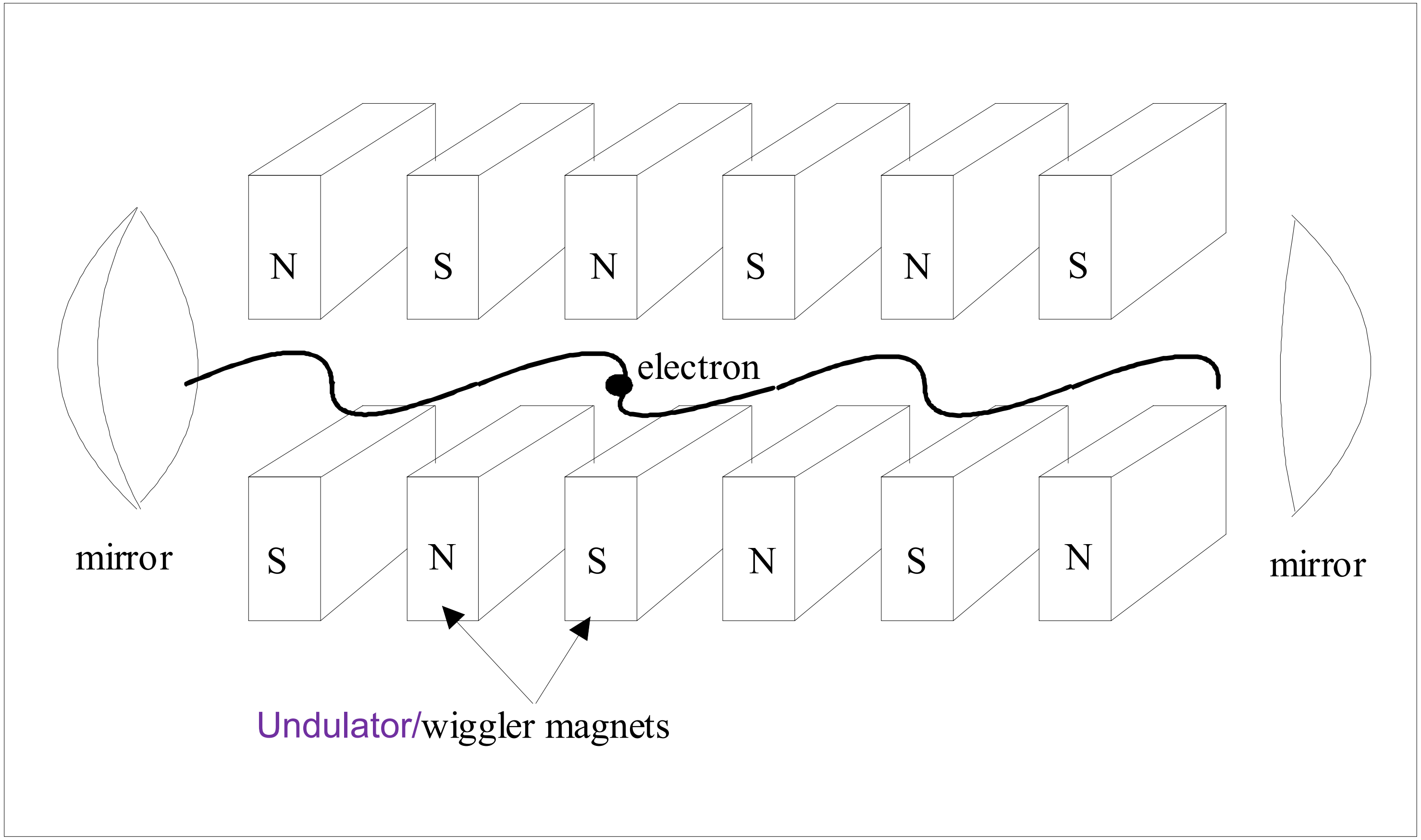
$$\Rightarrow \frac{1}{\gamma_z^2} = \frac{1+a_u^2}{\gamma^2} \quad \text{where} \quad a_u = 0.093 B_{rms}(\text{kgauss}) \times \lambda_u(\text{cm})$$

is called the *wiggler/ undulaotor parameter*

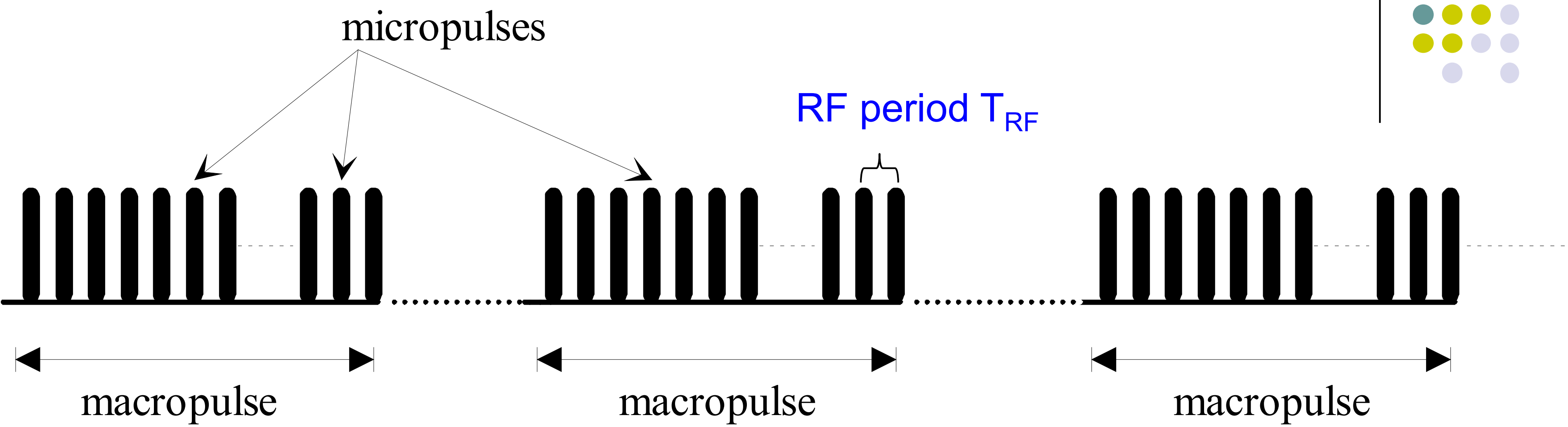
$$\Rightarrow \boxed{\lambda = \frac{1+a_u^2}{2\gamma^2}\lambda_u \quad \text{(FEL synchronism condition)}}$$

Undulator radiation wavelength can be tuned by magnetic field *B*, wiggler period λ_u, and electron energy γ

A Free-electron Laser Oscillator

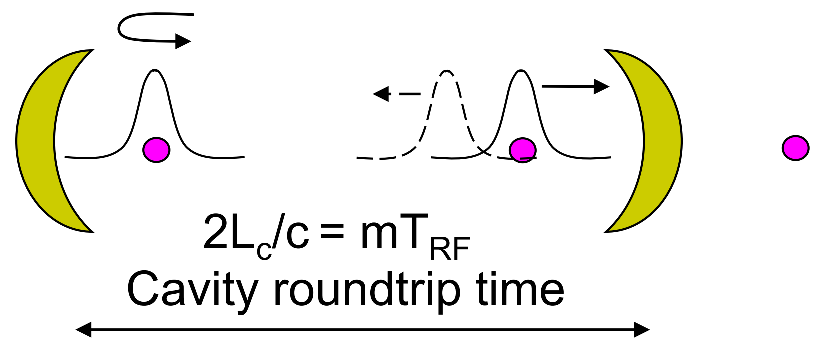


Pulse Structure of a RF Linac-driven FEL Oscillator



Taking SLAC S-band RF as an example:

1. RF frequency = 2.856 GHz
2. Micropulse length ~ 10 ps
3. Macropulse length: 1-5 μs
4. Macropulse repetition rate: 10-100 Hz



* Macropulse length > laser buildup time

Oscillation condition: $E_0 e^{-j\phi} e^{g_{th}L_c - 2\alpha L_c} = E_0$

$g_{th}L_c$: 1-way threshold gain, $2\alpha L_c$: roundtrip loss, ϕ = roundtrip phase

(1) Threshold condition: gain = loss $g_{th} = 2\alpha$ (2) Phase condition: $\phi = 2m\pi$

Electron-Wave Energy Exchange

$$\frac{dK}{dt} = e\vec{v}\cdot\vec{E}$$

K: electron kinetic energy

Wave Amplification $\Delta W = \int \vec{F}\cdot\vec{v}dt = e\int_{\tau=L/v_{//}} \vec{E}\cdot\vec{v}dt < 0$

Particle Acceleration $\Delta W = e\int_{\tau=L/v_{//}} \vec{E}\cdot\vec{v}dt > 0$

$v_{//}$: axial velocity
L: longitudinal length

Transverse Coupling
(Eg. Compton/Thomson/undulator radiation etc.)

$$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{\perp}\cdot\vec{v}_{\perp}dt$$

Longitudinal Coupling
(Eg. Smith-Purcell radiator, Traveling wave tube, backward-wave oscillator etc.)

$$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{//}\cdot\vec{v}_{//}dt$$

Resonant Interaction between Electron and Field

To have FEL gain

$$\boxed{\Delta W = e\int_{\tau=L_u/v_z} \vec{E}\cdot\vec{v}\,dt < 0}$$

L_u is the length of the undulator

For $E_x = E_0\cos(\omega t - kv_z t + \phi)$ and $v_x = \dfrac{-\sqrt{2}c_0 a_u}{\gamma}\cos(k_u v_z t)$

$$\Rightarrow \quad \vec{E}\cdot\vec{v} \propto \cos\{[k-(k+k_u)\beta_z]ct+\phi\} + \cos\{[k-(k-k_u)\beta_z]ct+\phi\}$$

Ψ:**pondermotive phase**

Whether $\vec{E}\cdot\vec{v} > 0$ (radiation) or $\vec{E}\cdot\vec{v} < 0$ (particle acceleration) depends on ϕ

To have appreciable value in $\int_{\tau=L_u/v_z} \vec{E}\cdot\vec{v}\,dt$

$$k-(k+k_u)\beta_z = 0 \quad \Rightarrow \quad \lambda = \frac{1+a_u^2}{2\gamma^2}\lambda_u$$

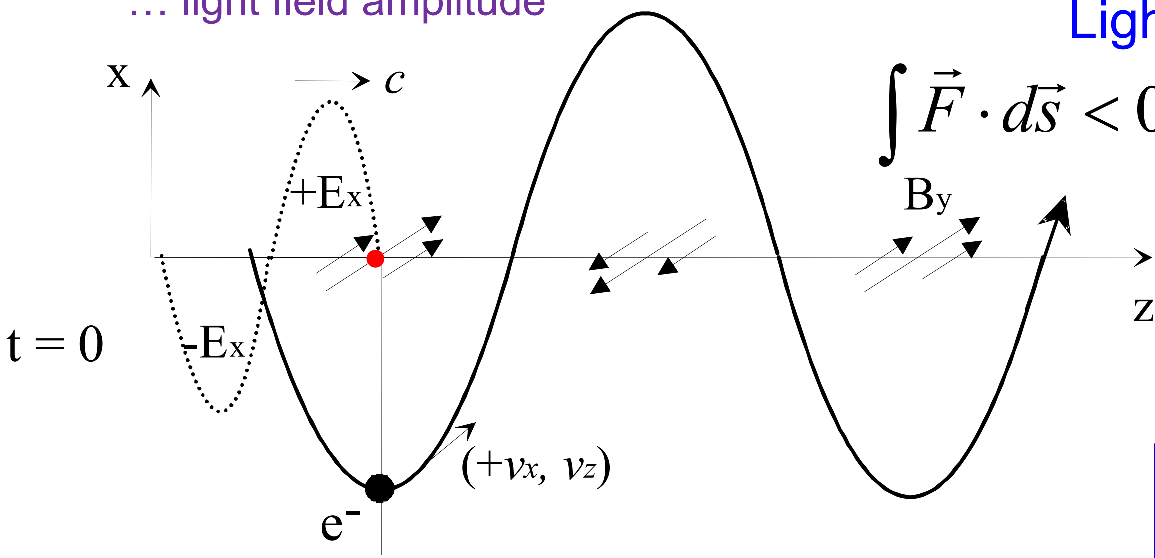
The FEL synchronism condition

$$k-(k-k_u)\beta_z = 0 \quad \Rightarrow \quad \beta_z \equiv v_z/c_0 > 1$$

Impossible in vacuum

Undulator Light Amplification

– Electron trajectory
… light field amplitude



Light amplification

$$\int \vec{F}\cdot d\vec{s} < 0 \text{ or } \int \vec{v}\cdot\vec{E}dt > 0$$

$$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{\perp}\cdot\vec{v}_{\perp}dt$$

$$\tau = \boxed{\frac{\lambda u/2 + \lambda_z/2}{c} = \frac{\lambda u/2}{vz}}$$

light slips one wavelength
ahead per wiggler period

$$\boxed{\lambda = \lambda_u\left(\frac{1}{\beta_z} - 1\right) \approx \frac{\lambda_u}{2\gamma_z^2}}$$

Pendulum Equation

The pondermotive (beat) phase $\psi = (k + k_u)z - \omega t$

was previously found from the beam-wave energy coupling equation

$$\frac{dK}{dt} = ev_x E_x = \frac{ec_0 a_u E_0}{\sqrt{2}\gamma} \cos\{\underbrace{\omega t - (k + k_u)z(t) + \phi}_{\psi}\}$$

power loss of an electron = time rate change of electron's kinetic energy

Take first derivative of ψ with respect to z and use the FEL synchronism condition $k - (k + k_u)\beta_{z,r} = 0$ to obtain

$$\frac{d\psi}{dz} = 2k_u \frac{\gamma - \gamma_r}{\gamma_r} = 2k_u \boxed{\frac{\Delta\gamma}{\gamma_r}}$$

where γ_r is the resonant particle energy satisfying the synchronism condition

$$\lambda = \lambda_u \frac{1 + a_u^2}{2\gamma_r^2} \quad \text{or} \quad k_u = k \frac{1 + a_u^2}{2\gamma_r^2}$$

A second derivative to the beat phase with respect to *z* gives the

pendulum equation

$$\boxed{\frac{d^2\psi}{dz^2} = -k_\psi^2 \sin\psi}$$

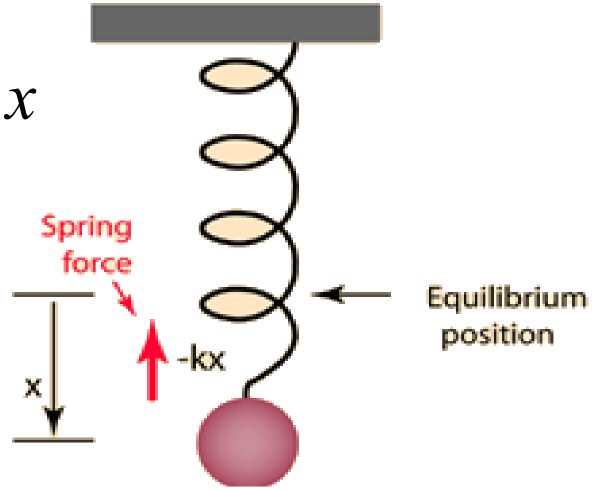
where $$k_\psi^2 = \left[\frac{e}{\gamma_r m_0 c_0}\right]^2 \frac{\sqrt{2} B_{rms} E_0}{c_0} \equiv \left(\frac{2\pi}{L_\psi}\right)^2$$

L_Ψ: synchrotron oscillation wavelength

For a small Ψ, $$\frac{d^2\psi}{dz^2} \sim -k_\psi^2 \psi$$

Recall the harmonic oscillator equation

$$\frac{d^2x}{dt^2} = -\omega_0^2 x$$

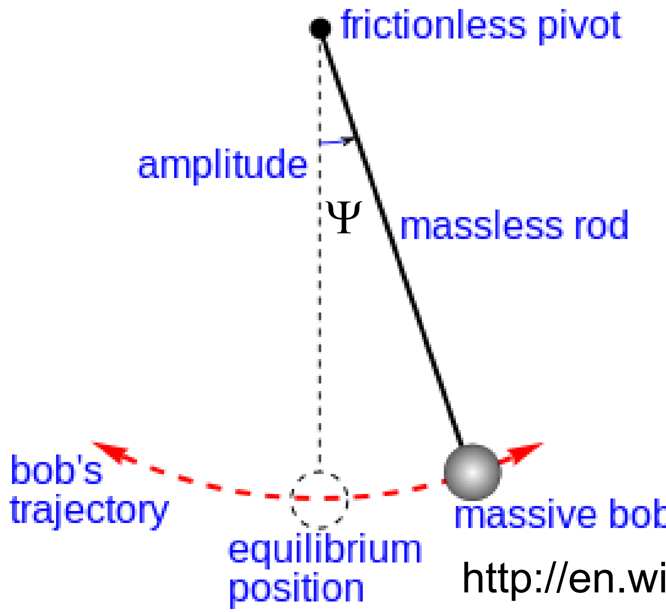


Particles oscillate, drift in the pondermotive phase .

http://hyperphysics.phy-astr.gsu.edu/hbase/oscda.html

pendulum equation

$$\boxed{\frac{d^2\psi}{dz^2} = -k_\psi^2 \sin\psi}$$

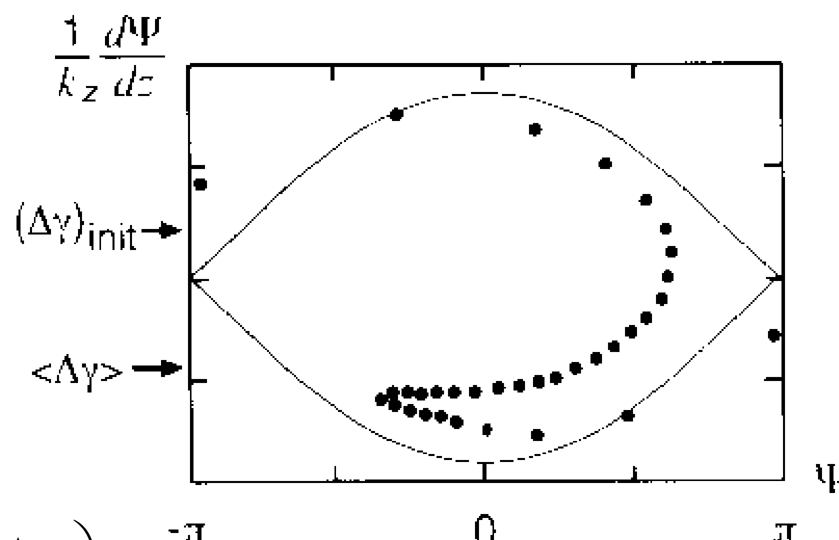


http://en.wikipedia.org/wiki/Pendulum

With the definition of k_ψ, the *phase diagram* can be plot from

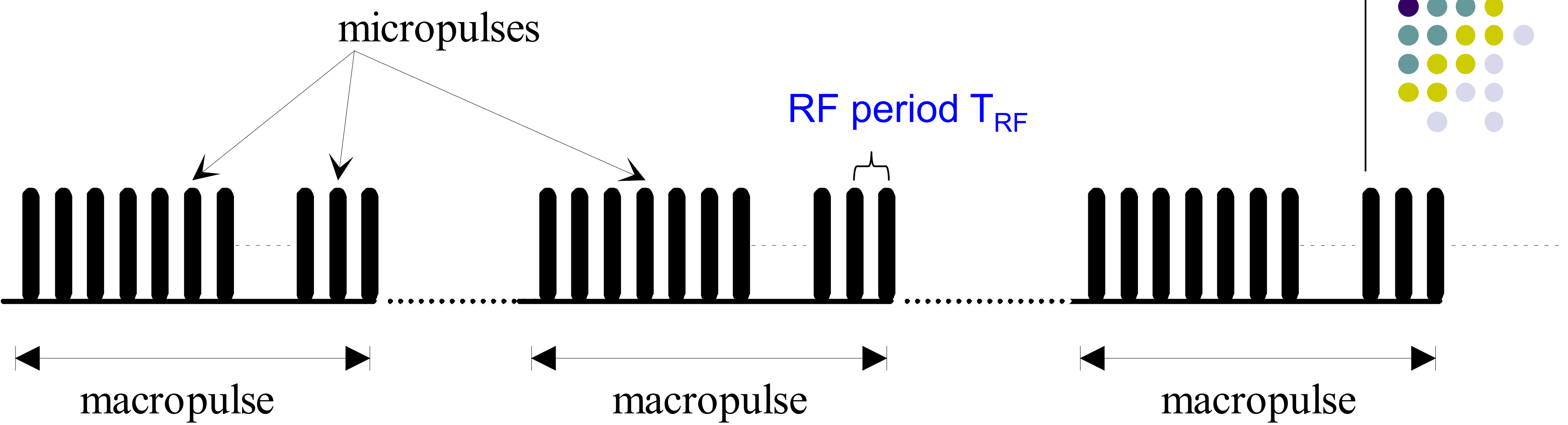
$$\frac{d\psi}{dz} = \pm\sqrt{2}k_\psi\sqrt{\cos\psi + 1} = 2k_u\frac{\Delta\gamma}{\gamma_r}$$

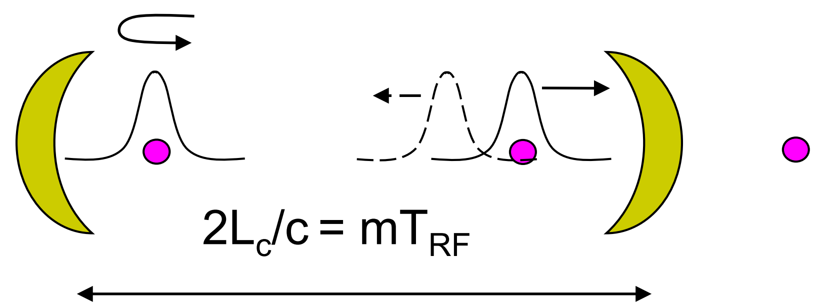
The bucket height = 4 k_ψ, and the maximum energy extraction occurs at half synchrotron wavelength: FEL length is ~$L_\psi/2$



The maximum energy efficiency for an FEL = $\left(\frac{\Delta\gamma}{\gamma_r}\right)_{\max} = 1/(2N_u)$

FEL Buildup Time τ_B





Saturation power

Roundtrip net gain

$$\underbrace{P_b \times \left[\frac{1}{2N_w}\right]}_{\text{Saturation power}} = P_{sp} \times \left[\underbrace{e^{(gL_c - 2\alpha L_c)}}_{\text{Roundtrip net gain}}\right]^{\frac{\tau_B}{2L_c/c}}$$

\# of roundtrips

Beam power

Spontaneous radiation power

$$\left(\frac{\Delta\gamma}{\gamma_r}\right)_{\max} = 1/(2N_w)$$

* Macropulse length > laser buildup time

$\boxed{\tau_M > \tau_B}$

FEL Gain

$$G = \frac{W_f - W_i}{W_i} = e^{gL_c}$$

To have gain

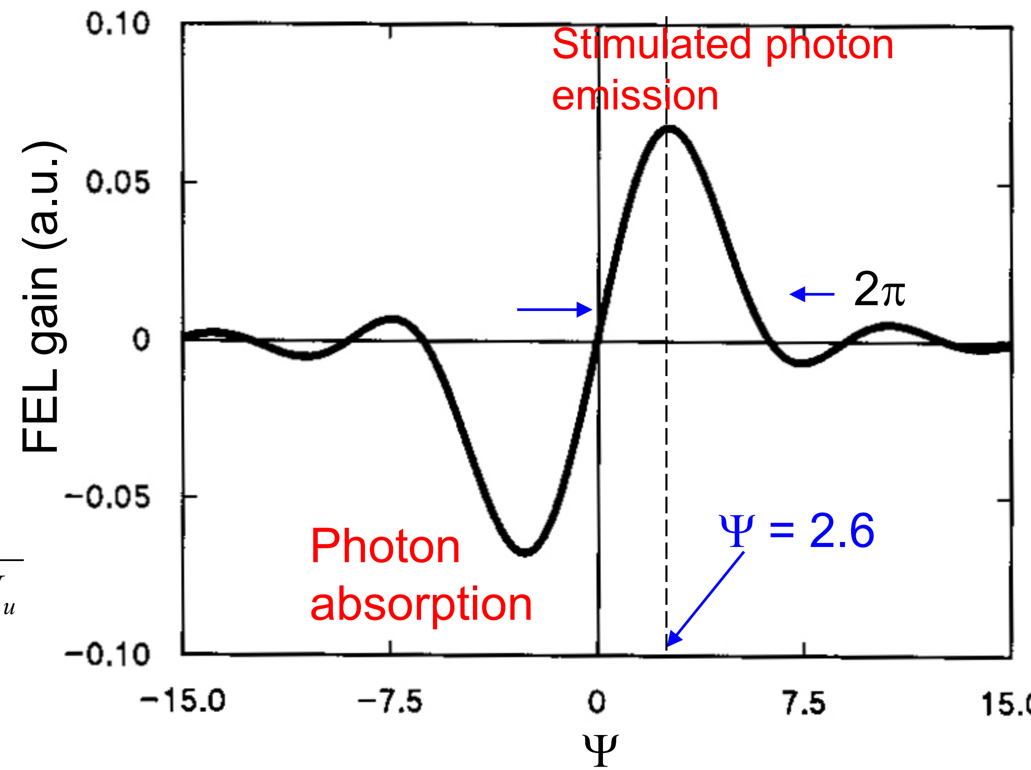
$$\Delta W = e\int_{\tau = L_u / v_z} \vec{E}\cdot\vec{v}\, dt < 0$$

L_u is the length of the undulator

For $E_x = E_0 \cos(\omega t - k v_z t + \phi)$ and $v_x = \frac{-\sqrt{2}ca_u}{\gamma}\cos(k_u v_z t)$

whether $\vec{E}\cdot\vec{v} > 0$ (radiation) or $\vec{E}\cdot\vec{v} < 0$ (particle acceleration) depends on ϕ

- **Gain is small for short wavelength FEL**
- **Electron injection energy has to be detuned from synchronism (slightly larger)**
- **Gain is proportional to injection current**
- **Energy spread can't exceed** $\frac{\Delta\gamma}{\gamma} < \frac{1}{2N_u}$ **where N_u is the number of wiggler periods**

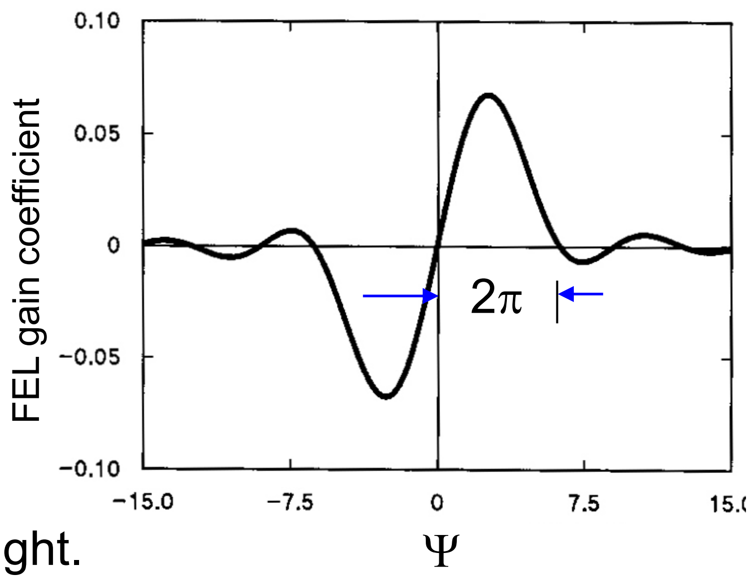


Energy Spread Requirement

Refer to the FEL gain curve, for an electron to contribute its energy to the FEL gain, the acceptance phase width has to be confined to 2π or

$$\Delta\Psi = \left| [\omega - (k+k_u)\bar{v}_z]\frac{L_u}{\bar{v}_z} \right|_{\max} = 2\pi \Rightarrow \left| [\frac{\omega}{\bar{v}_z} - (k+k_u)]L_u \right|_{\max} = 2\pi$$

$$\Rightarrow \left| \frac{d\psi}{dz} L_u = 2k_u \frac{\Delta\gamma}{\gamma_r} L_u \right| < 2\pi \Rightarrow \boxed{\left|\frac{\Delta\gamma}{\gamma_r}\right| < \frac{1}{2N_u}}$$



where N_u is the number of undulator periods

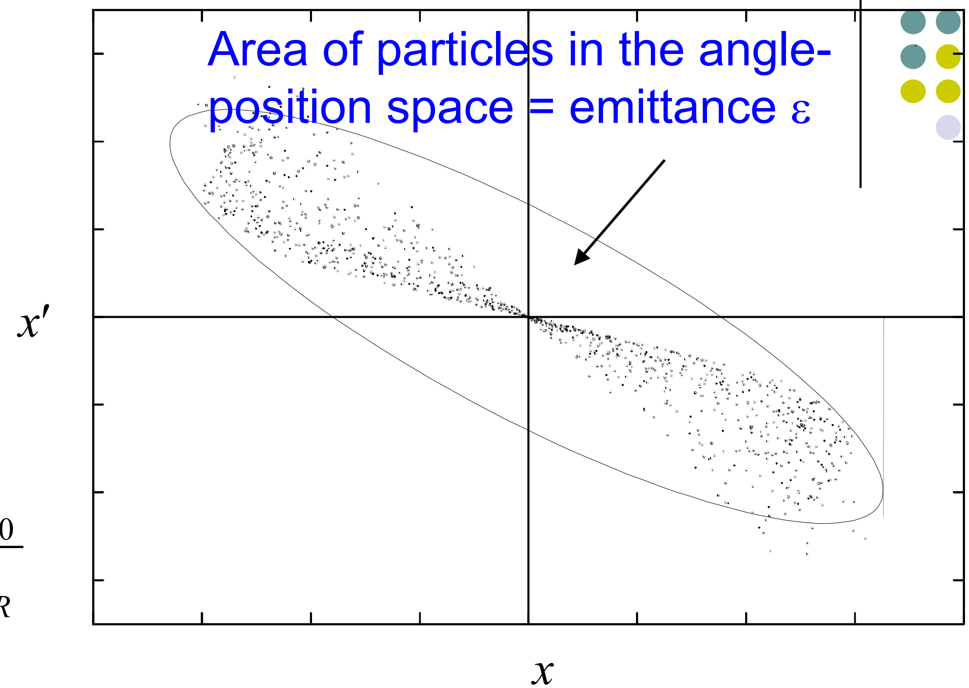
* This result is already known from the bucket height.

So, the energy spread of the electron beam for an FEL has to be less than $1/(2N_u)$

Emittance Requirement for an FEL

An Electron Beam

The phase (angle-position) space area of electrons is called the beam's geometric emittance ε



A Gaussian Laser Beam

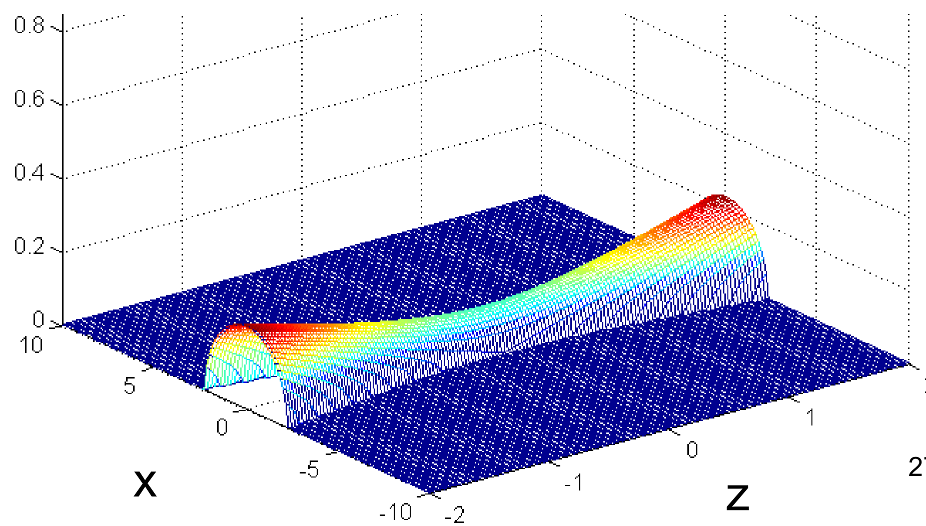
Rayleigh range $z_{o,R} = \dfrac{\pi w_0^2}{\lambda}$

Far-field diffraction angle = $\theta \sim \dfrac{w_0}{z_R}$

The phase space (angle and laser beam size) area is $\pi \times \theta \times w_0 \sim \lambda$

To place an electron beam
Inside an optical beam

$\varepsilon < \lambda$



Therefore long-wavelength FEL is more forgiving to electron-beam quality

FEL Gain Bandwidth

The spectral bandwidth is defined by the variation of the spectral ratio

$$\left|\frac{\Delta\lambda}{\lambda}\right| = \left|\frac{\Delta\omega}{\omega}\right|$$

within the half width of the gain curve

$$\Delta\Psi = \left|\Omega\tau = [\omega - (k + k_u)\bar{v}_z]\frac{L}{\bar{v}_z}\right| < \pi$$

From the FEL synchronism condition $\lambda = \lambda_u \dfrac{1 + a_u^2}{2\gamma^2}$, it is straightforward to show

$$\left|\frac{\Delta\lambda}{\lambda}\right| = 2\left|\frac{\Delta\gamma}{\gamma}\right|$$

However the maximum allowed $\Delta\gamma/\gamma < 1/(2N_w)$ is obtained from the full width. For a half width

$$\left|\frac{\Delta\lambda}{\lambda}\right| = 2\left|\frac{\Delta\gamma}{\gamma}\right| < 2 \times \frac{1}{2N_u} \times \frac{1}{2} = \boxed{\frac{1}{2N_u}}$$

Characteristics of an Undulator FEL

1. Laser: a coherent light source

2. Wavelength tunable:
   by varying the magnetic field and the electron energy

3. High peak power: GW-MW in 0.1~ 10 psec micropulse

4. High average power: kW in > ~ μsec macropulse

General Requirements for Building an FEL Oscillator

Gain > loss

In particular

i. Electron energy spread $\Delta\gamma/\gamma < 1/2N_u$

ii. Electron emittance $\varepsilon < \lambda$

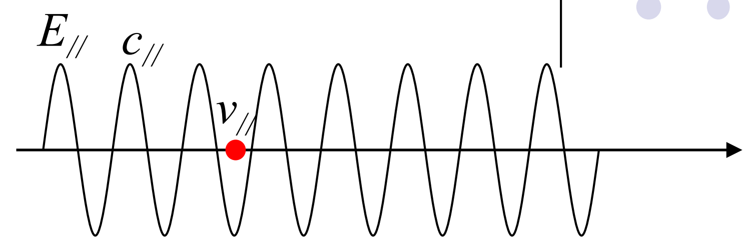
Part II – Slow-wave FEL

Cherenkov, Traveling-wave Tube (TWT), Backward-wave Oscillator (BWO), Smith-Purcell FEL

Longitudinal Energy Coupling

(Eg. Smith-Purcell radiator, traveling-wave tube, backward-wave oscillator etc.)

$$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{//} \cdot \vec{v}_{//}\, dt$$



For ΔW to accumulate, it requires

$$c_{//} = v_{//} \qquad (1)$$

where $v_{//}$ is the electron's longitudinal velocity and $c_{//}$ is the speed of an EM wave along the same direction.

But the vacuum speed of an EM wave c_0 is always larger than electron moving speed, and thus

$$c_0 > v_{//} \qquad (2)$$

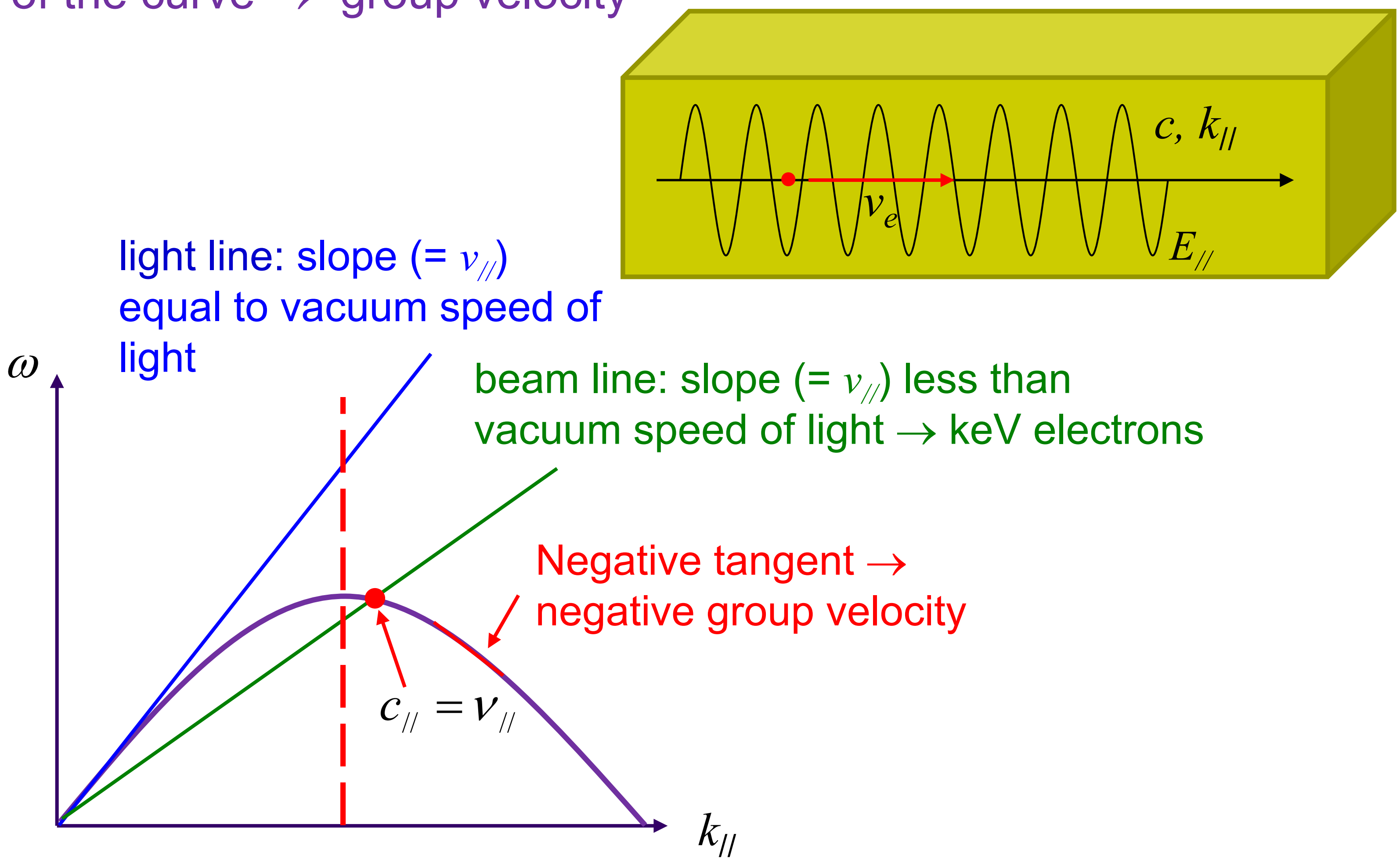
Eq. (1) is possible only for two situations:

1. **The EM wave is in a medium with refractive index *n* > 1**
2. **The EM wave is an evanescent/slow wave**

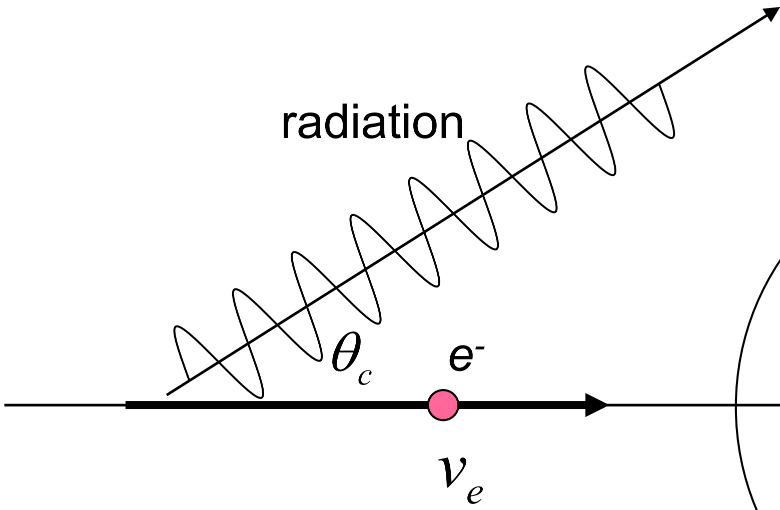
Dispersion Diagram with a Beam Line

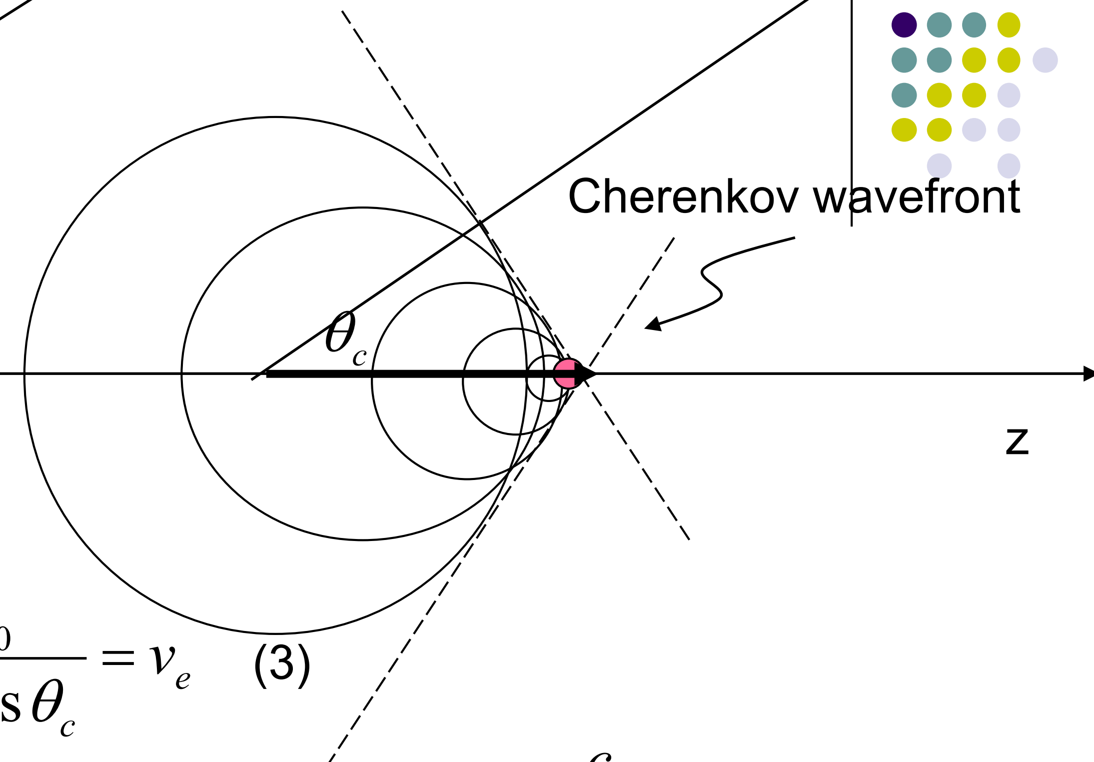
—— dispersion curve of a slow-wave electromagnetic structure

- Slope of the curve → phase velocity $c_{//}$
- Tangent of the curve → group velocity



1. In a medium with refractive index = n





The condition $c_{//} = v_{//}$

means $$c_z = \frac{\omega}{k_z} = \frac{\omega}{k\cos\theta_c} = \frac{c_0}{n\cos\theta_c} = v_e \qquad (3)$$

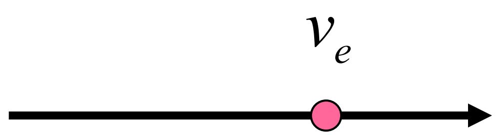
Since $c_0 > v_e$, n has to be larger than 1 to obtain to result $\dfrac{c_0}{n\cos\theta_c} = v_e$ (4)

This result also means that the electron velocity has to be larger than the EM wave velocity to have Cherenkov radiation or

$$v_e > \frac{c_0}{n} = c \qquad \text{(Cherenkov threshold)}$$

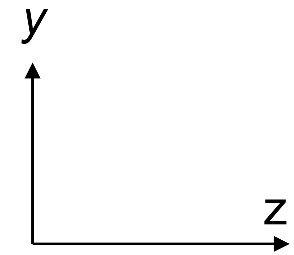
The angle $\theta_c = \cos^{-1}(\dfrac{1}{n\beta})$ is called the Cherenkov angle. (5)

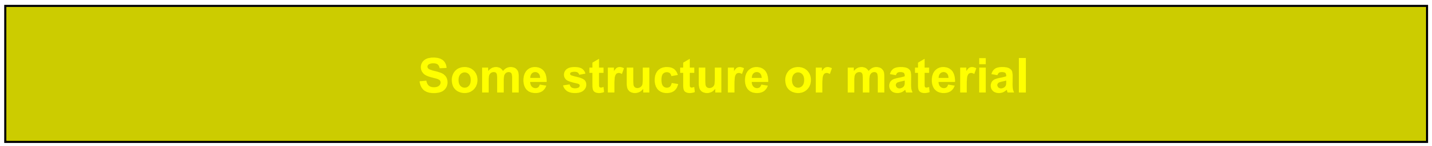
2. The EM wave is an evanescent wave



$$E_z \sim e^{-jk_z z - \alpha y}$$

Evanescent/slow wave





Assume no variation along x, substitute $E \sim e^{-jk_z z - \alpha y}$ into the Helmholtz equation

$\nabla^2 E_z + k_0^2 E_z = 0$ to obtain $\boxed{k_0^2 = k_z^2 - \alpha^2.}$

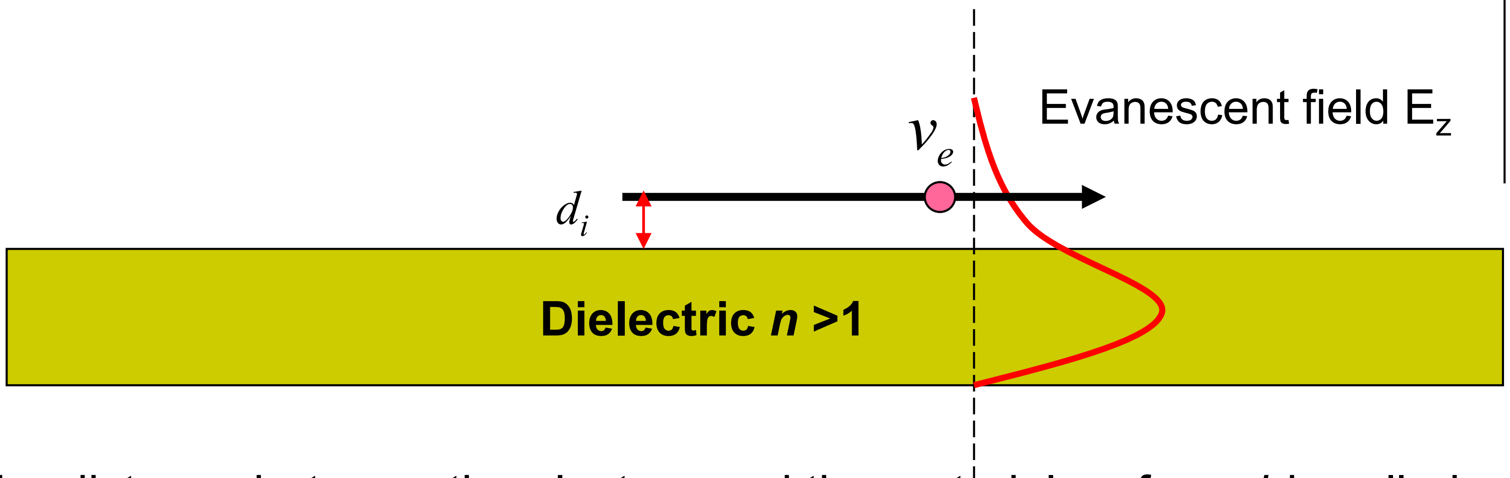
The condition $c_{//} = v_{//}$ means $c_z = \dfrac{\omega}{k_z} = \dfrac{\omega}{\sqrt{k_0^2 + \alpha^2}} = \dfrac{c_0}{\sqrt{1 + \alpha^2 / k_0^2}} = v_e$ (6)

Note that $c_0 > v_e$. By properly designing α (through the structure or material), it is possible to match the condition $c_{//} = v_{//}$

For $\gamma >> 1$, Eq. (6) further reduces to $\alpha = \dfrac{2\pi}{\beta\gamma\lambda_0}$ or $\dfrac{1}{\alpha} = \dfrac{\beta\gamma\lambda_0}{2\pi}$, (7)

where λ_0 is the radiation wavelength in vacuum.

Impact Parameter



The distance between the electron and the material surface d_i is called the *impact parameter*.

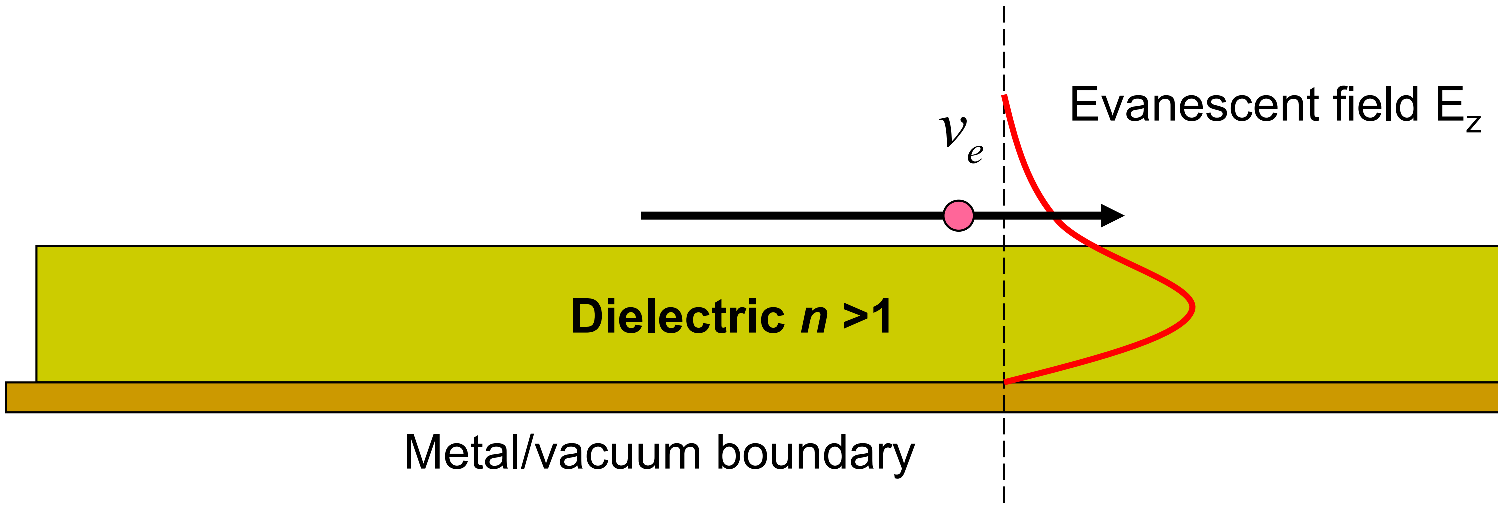
To have appreciable coupling in $\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{//}\cdot\vec{v}_{//}dt$, one will certainly position

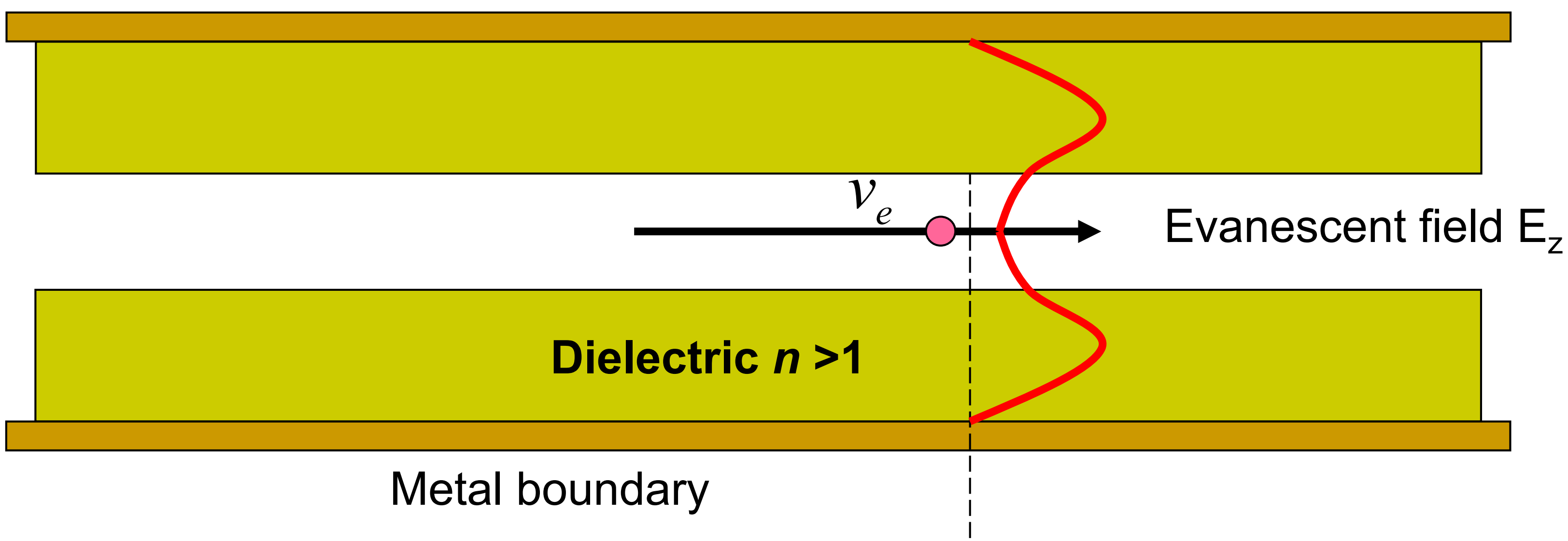
the electron beam within a transverse distance < 1/α or

$$d_i < \frac{1}{\alpha} = \frac{\beta\gamma\lambda_0}{2\pi}$$

This condition tells "**long radiation wavelength and high energy beam**" will make the alignment much easier for this kind of radiation device (Cherenkov & Smith-Purcell).

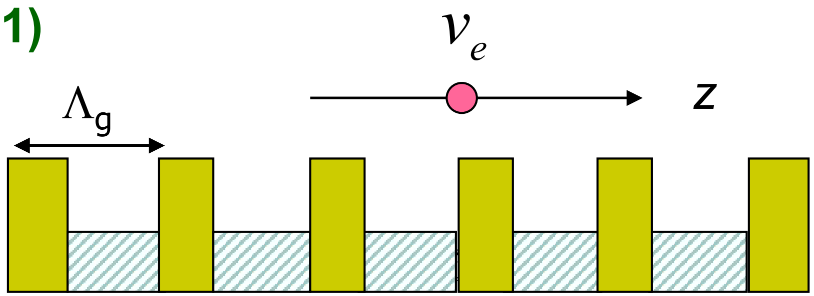
Variants of Cherenkov Radiation





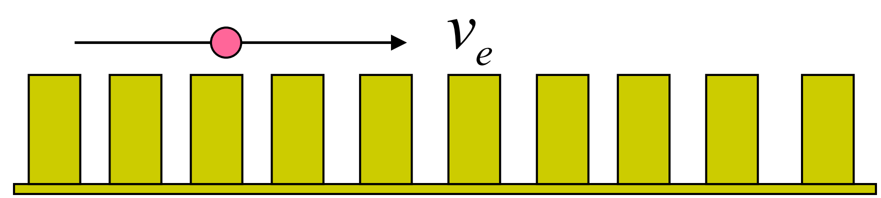
Traveling-wave Tube (TWT) /Backward-wave Oscillator (BWO)

(1)

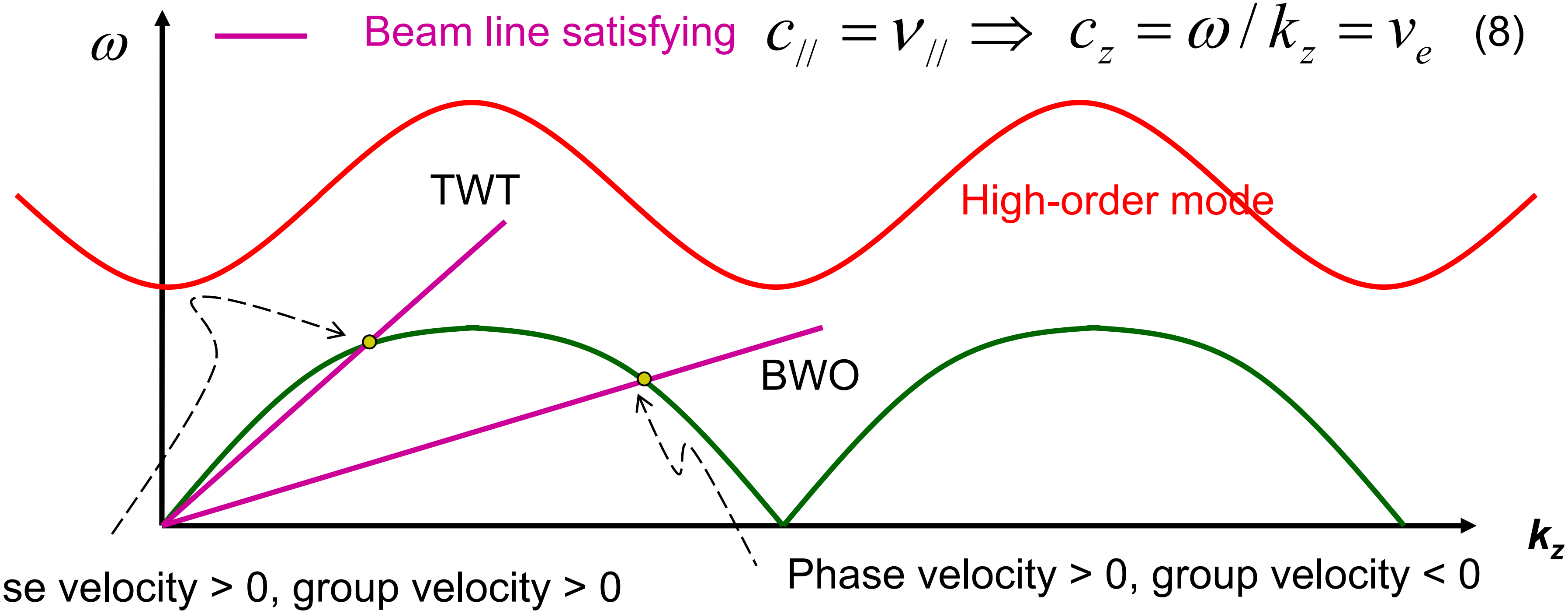


A grating is a slow-wave structure supporting a slow **surface wave**.

(2)



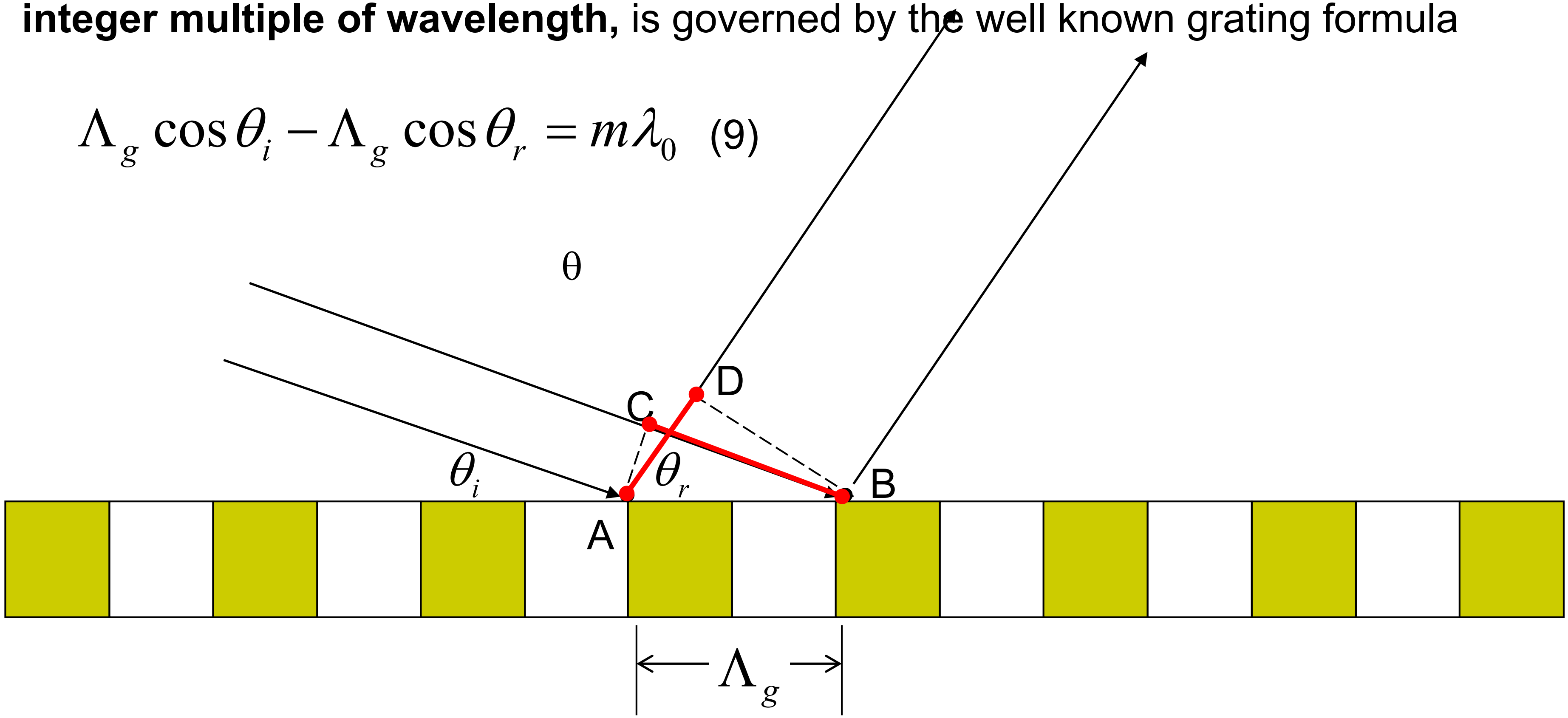
With a grating structure, the dispersion curve bends.



However, a grating (periodical structure) is not just a slow-wave structure , but **a resonant structure!**

The condition for the constructive interference in the far-field zone, **BC-AD = integer multiple of wavelength,** is governed by the well known grating formula

$$\Lambda_g \cos\theta_i - \Lambda_g \cos\theta_r = m\lambda_0 \quad (9)$$



For the special case of $\theta_i = 0, \theta_r = 180^o$ (reflection resonance) and m = 1, (10)

it gives the well known **distributed feedback resonance** in modern optics $\Lambda_g = \dfrac{\lambda_0}{2}$

Smith Purcell Radiation: grating resonant radiation

With $\tau = \dfrac{\Lambda_g}{v_e} = \dfrac{\Lambda_g}{c_0 \beta}$, constructive interference requires

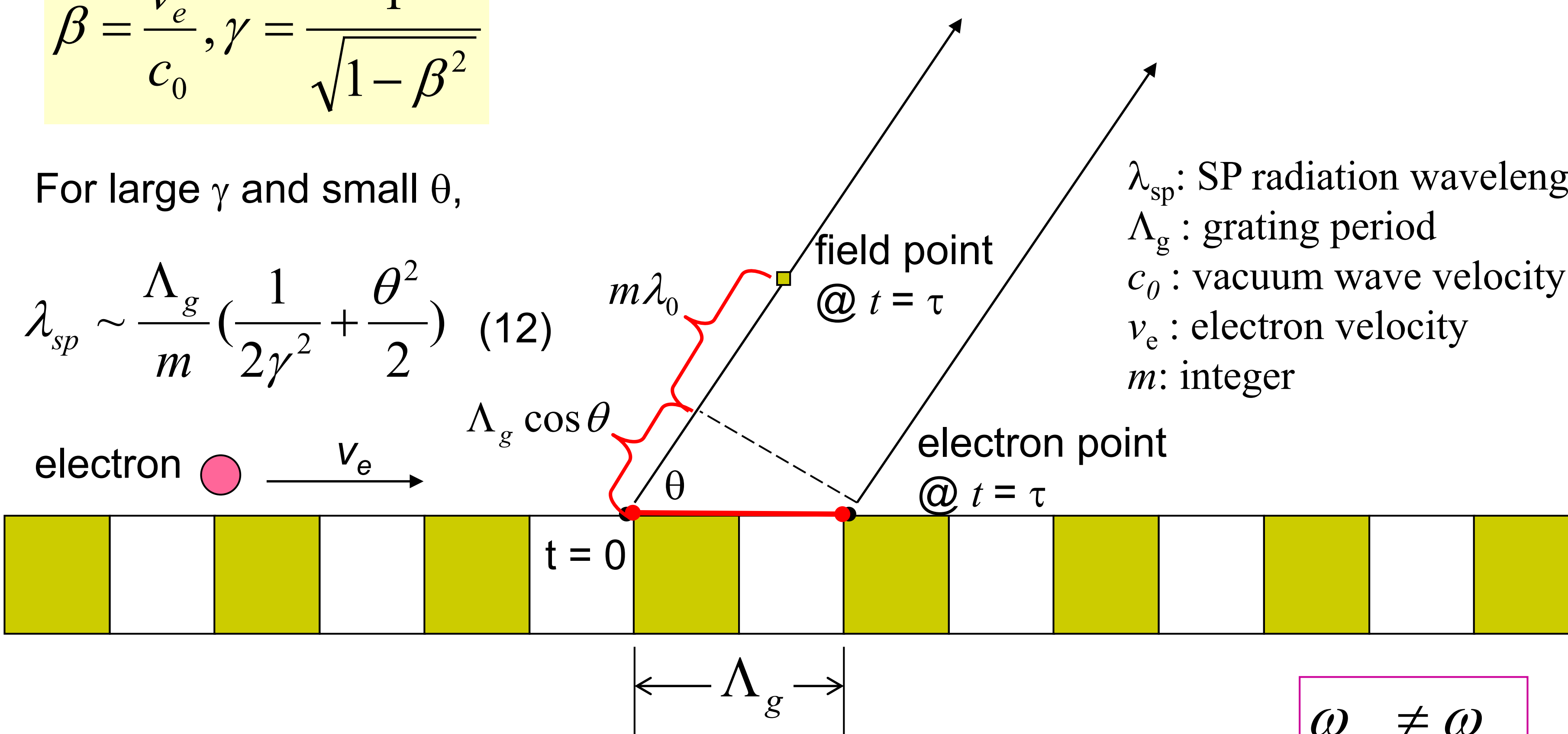
$$c_0 \cdot \tau - \Lambda_g \cos\theta = m\lambda_{sp} \Rightarrow \lambda_{sp} = \frac{\Lambda_g}{m}\left(\frac{1}{\beta} - \cos\theta\right) \quad (11)$$

$$\beta = \frac{v_e}{c_0}, \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Smith-Purcell radiation condition

For large γ and small θ,

$$\lambda_{sp} \sim \frac{\Lambda_g}{m}\left(\frac{1}{2\gamma^2} + \frac{\theta^2}{2}\right) \quad (12)$$



λ_{sp}: SP radiation wavelength
Λ_g : grating period
c_0 : vacuum wave velocity
v_e : electron velocity
m: integer

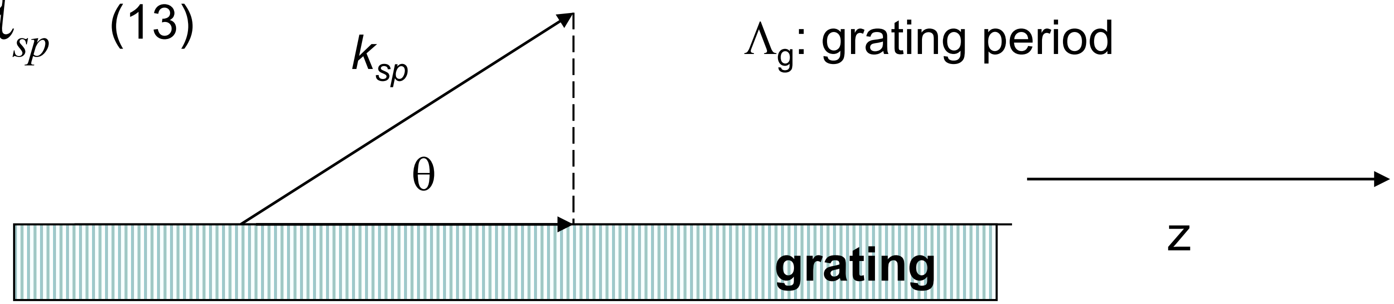
$$\boxed{\omega_{sp} \neq \omega_{sw}}$$

Note that the Smith-Purcell and slow-wave frequencies are different

In general $\omega_{sp} > \omega_{sw}$

The Smith-Purcell radiation wavelength λ_{sp} is shorter than the slow-wave (TWT/BWO) wavelength λ_{sw} for a given electron energy. This is an **advantage** of an SP FEL.

Proof of $\lambda_{sw} > \lambda_{sp}$ (13)



To have energy coupling, the SP mode is phase matched to the SW mode on the surface

Matching k along z: $\boxed{k_{sp,z} = k_{sp}\cos\theta = k_{sw,z}}$ (14) and $k_{sp} = \dfrac{2\pi}{\lambda_{sp}}$. (15)

But, for a slow wave, $k_0^2 = k_z^2 - \alpha^2$, $\rightarrow k_{sw,z}^2 = k_{sw}^2 + \alpha^2$ (16)

Combine (14) and (16) to obtain $k_{sp}^2\cos^2\theta = k_{sw}^2 + \alpha^2 \rightarrow k_{sp}^2 = \dfrac{k_{sw}^2}{\cos^2\theta} + \dfrac{\alpha^2}{\cos^2\theta}$

Since $\cos^2\theta \le 1, \alpha > 0$, one has $k_{sp}^2 > k_{sw}^2$ or $\lambda_{sw} > \lambda_{sp}$. (17)

Wave-electron energy exchange (longitudinal coupling)

Start with a single particle. The rate of loss or gain of electron's kinetic energy is

$$\frac{dK}{dt} = \vec{F} \cdot \vec{v} = -e v_z E_z \,, \qquad (18)$$

where the electron kinetic energy and the E_z field are

$$K = (\gamma - 1) m_0 c_0^2 \,, E_z = E_0 \cos(\omega t - k_z z + \phi) \qquad (19)$$

Define the phase $\psi = k_z z - \omega t - \phi + \dfrac{\pi}{2}$ to obtain the **pendulum equation**

$$\boxed{\frac{d^2 \psi}{dz^2} = -k_\psi^2 \sin \psi} \qquad (20)$$

where $k_\psi = \sqrt{\dfrac{\omega e E_0}{\gamma_r^3 v_r^3 m_0}} \equiv \dfrac{2\pi}{L_\psi}$ (21) with the resonant electron speed $v_r = \dfrac{\omega}{k_z}$

and the resonant electron energy $\gamma_r = \dfrac{1}{\sqrt{1 - v_r^2 / c_0^2}}$ (22) , and L_ψ the so-called "synchrotron wavelength".

The rest is the same as the discussion for the fast-wave (undulator) FEL.