INTRODUCTION TO HIGH BRIGHTNESS ELECTRON BEAMS

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NSRRC Winter School on Free Electron Lasers 2019

Jan. 14th – 18th, 2019

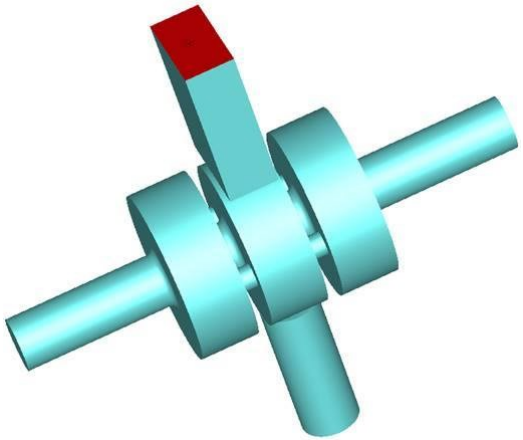
Single Particle Dynamics in EM Fields

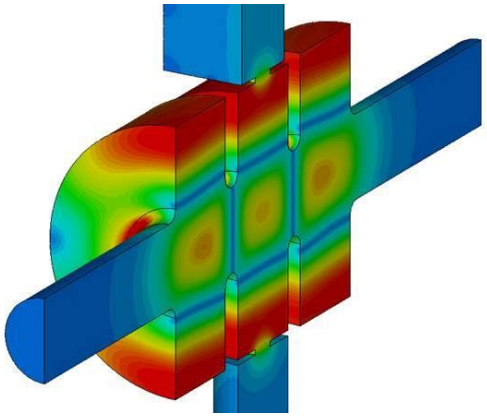
$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma m_0 \vec{v}\right) = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Lorentz force law

The Lorentz factor γ is in general not a constant of time

- Relativistic dynamics
- Applied EM fields are usually expressed as expanded series called paraxial approximation
- Complicated EM field distributions can only be solved by numerical methods. Examples of simulation codes are:
  - POISSON/SUPERFISH
  - High Frequency Structure Simulator (HFSS) -- http://www.ansoft.com/products/hf/hfss/
  - CST Microwave Studio -- http://www.cst.com/Content/Products/MWS/Overview.aspx

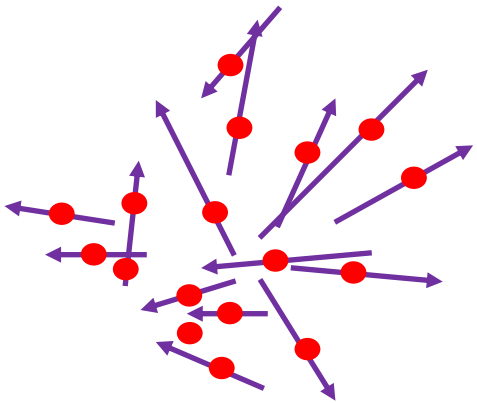




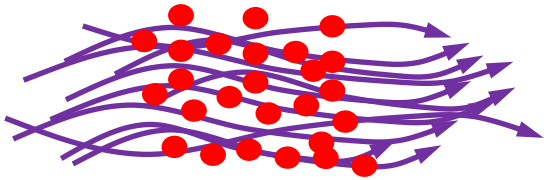
What is a Charged Particle Beam?

an “ordered flow” of charged particles

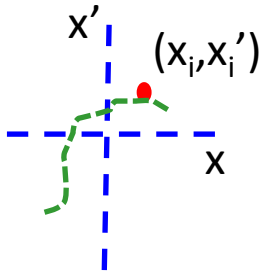
all particles are moving
along the same trajectory
for a perfect beam



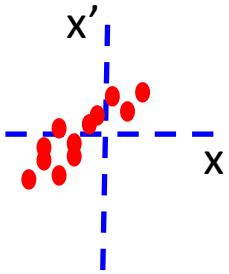
a random distribution
of charges



something in between
(real world)



a single particle
in trace-space



a system of particles
in trace-space

Trace Space Area Definition of Emittance

- Trace space area definition of emittance

$$\varepsilon_x = A_x = \iint dxdx' \qquad [\pi \text{ m-rad}]$$

many authors identify the emittance as trace-space area divided by π (unit: [m-rad])

- The trace-space area A_x is related to the phase-space area in x-p_x plane by

$$A_x = \frac{1}{\langle p_z \rangle} \iint dxdp_x = \frac{1}{\gamma\beta mc} \iint dxdp_x$$

this integral can be a constant as long as Liouville's theorem applies.

- We can therefore useful to define normalized emittance that is independent of particle acceleration such that

$$\boxed{\varepsilon_n = \beta\gamma\varepsilon}$$

- However, beams with quite different distributions in trace space may have the same area!!

Beam Size and Beam Divergence

Let $f(x,x')$ be the distribution function such that $\int f(x,x')dxdx' = N$. N is the total number of particles. Beam parameters can be defined accordingly as:

$$\langle x \rangle = \int x f(x,x')dxdx' / \int f(x,x')dxdx'$$ **averaged beam size**

$$\langle x' \rangle = \int x' f(x,x')dxdx' / \int f(x,x')dxdx'$$ **averaged beam divergence**

$$\sigma_x = \sqrt{\int (x - \langle x \rangle)^2 f(x,x')dxdx' / \int f(x,x')dxdx'}$$ **rms beam size**

$$\sigma_{x'} = \sqrt{\int (x' - \langle x' \rangle)^2 f(x,x')dxdx' / \int f(x,x')dxdx'}$$ **rms beam divergence**

$$\sigma_{xx'} = \sqrt{\int (x - \langle x \rangle)(x' - \langle x' \rangle) f(x,x')dxdx' / \int f(x,x')dxdx'}$$ **beam correlation**

root-mean square of a set of n values is defined as:

$$\boxed{x_{rms} = \sqrt{\frac{1}{n}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)}}$$

Beam Emittance

- Define rms emittance as

$$\sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{bmatrix}$$

$$\tilde{\varepsilon}_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \det \sigma$$

σ-matrix of beam distribution

- If x and x' are not correlated (at beam waist where the beam is neither converging nor diverging)

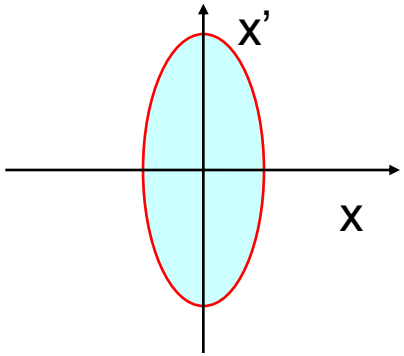
$$\tilde{\varepsilon}_x = \tilde{x}\tilde{x}' = \sigma_x \sigma_{x'}$$

- RMS emittance gives *quantitative* information on beam quality
- RMS emittance gives more weight to the particles in the outer region of the trace-space area. Therefore, remove some of the outer particles will significantly improve RMS emittance without too much degradation of beam intensity.

Effective Emittance

- In a system that all forces (space charge and external forces) acting on the particles are linear (i.e. proportional to particle displacement x from the beam axis), it is still ***useful*** to assume an elliptical shape for the area occupied by the beam in trace space such that

$$A_x = \pi x_m \left(x'_m\right)$$



- We are able to define an emittance as

$$\varepsilon_x = x_m \left(x'_m\right) = \frac{A_x}{\pi}$$

- The relation between $x_m, \left(x'\right)_m, \varepsilon_x$ and the corresponding RMS quantities $\tilde{x}, \tilde{x}'_{th}, \tilde{\varepsilon}_x$ are given by

$$x_m = 2\tilde{x}$$

$$\left(x'\right)_m = 2\tilde{x}'_{th}$$

$$\varepsilon_x = 4\tilde{\varepsilon}_x$$

Beam Brightness

- Definition of beam brightness:

$$B = \frac{J}{d\Omega} = \frac{dI}{ds\,d\Omega}$$

- It is useful know that total beam current that can be confined within a 4-D trace space volume V_4. We can define average brightness as

$$\overline{B} = \frac{I}{V_4} \qquad V_4 = \iint ds\,d\Omega$$

- If any particle distribution whose boundary in 4-D trace space is defined by a hyperellipsoid

$$\frac{x^2}{a^2} + \left(\frac{ax'}{\varepsilon_x}\right)^2 + \frac{y^2}{b^2} + \left(\frac{by'}{\varepsilon_y}\right)^2 = 1 \qquad \overline{B} = \frac{2I}{\pi^2 \varepsilon_x \varepsilon_y}$$

one finds $\iint ds\,d\Omega = \left(\pi^2/2\right)\varepsilon_x \varepsilon_y$ and average brightness is

- RMS brightness is then defined as:

$$\boxed{\widetilde{B} = \frac{I}{\widetilde{\varepsilon}_x \widetilde{\varepsilon}_y}}$$

(note: $2/\pi^2$ is left out)

Thermal Emittance of a Beam from Cathode

- For a beam from a thermionic cathode at temperature T, rms thermal velocity spread is related to rms beam divergence such that

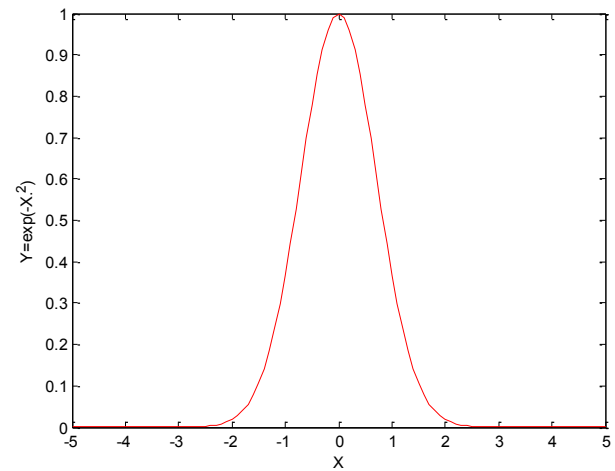
$$\tilde{x}' = \tilde{v}_{x,th} / v_0$$

- Assume Maxwellian velocity distribution from a round cathode with radius r_s,

$$f(v_x, v_y, v_z) = f_0 \exp\left[-\frac{m\left(v_x^2 + v_y^2 + v_z^2\right)}{2k_B T}\right]$$

where T is the temperature of the cathode, then

$$\tilde{v}_x = \tilde{v}_y = \left(\frac{k_B T}{m}\right)^{1/2}$$



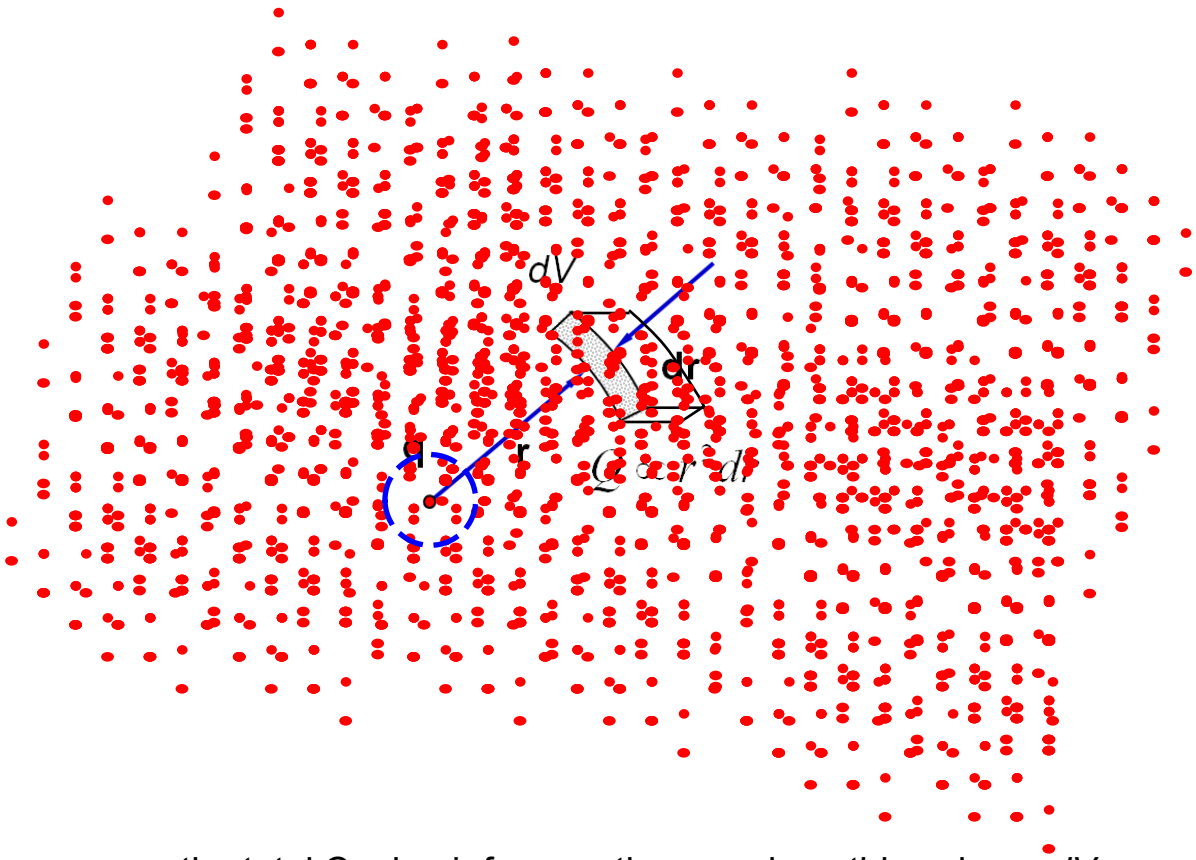
- On the other hand, for a beam emitted from the cathode

$$\tilde{x} = \tilde{y} = r_s / 2$$

$$\tilde{x} = \frac{\tilde{r}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[\frac{2\pi n_0 \int_0^a r^3 dr}{2\pi n_0 \int_0^a r dr}\right]^{1/2}$$

- Thermal emittance of such beam from the cathode is given by

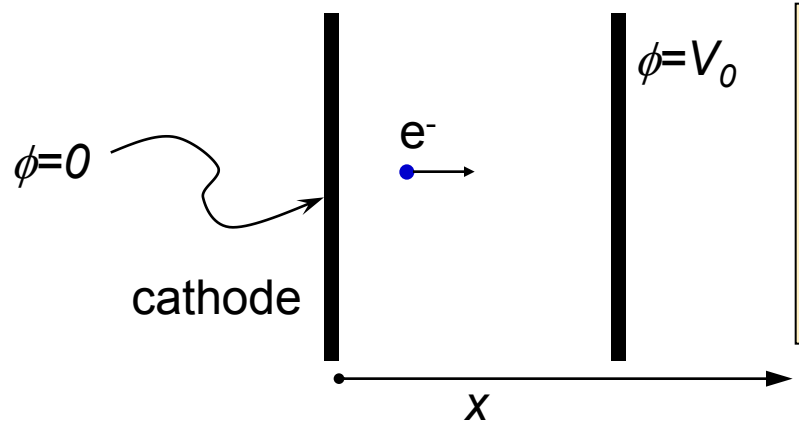
$$\tilde{\varepsilon}_{x,y} = 2r_s \frac{\left(\frac{k_B T}{m}\right)^{1/2}}{v_0} \rightarrow \tilde{\varepsilon}_n = 2r_s\left(\frac{k_B T}{mc^2}\right)^{1/2}$$



the total Coulomb force acting on *q* by a thin volume *dV* of charge Q is independent of *r* !!

Planar Diode with Space Charge – Child-Langmuir Law

$$\begin{cases} \nabla^2\phi = \dfrac{d^2\phi}{dx^2} = -\dfrac{\rho}{\varepsilon_0} \\ J_x = \rho\dot{x} = const \\ \dfrac{m}{2}\dot{x}^2 - e\phi(x) = 0 \end{cases}$$



Emission of electrons from cathodes:

1. Thermionic emission
2. Secondary emission
3. Photo-emission

$$\Rightarrow \frac{d^2\phi}{dx^2} = \frac{J}{\varepsilon_0 (2e/m)^{1/2}} \frac{1}{(\phi)^{1/2}}$$

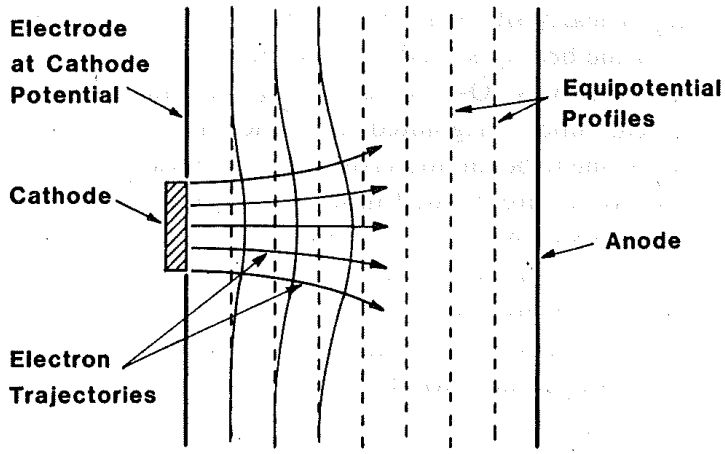
$$\Rightarrow \left(\frac{d\phi}{dx}\right)^2 = \frac{4J}{\varepsilon_0 (2e/m)^{1/2}} \phi^{1/2} + C$$

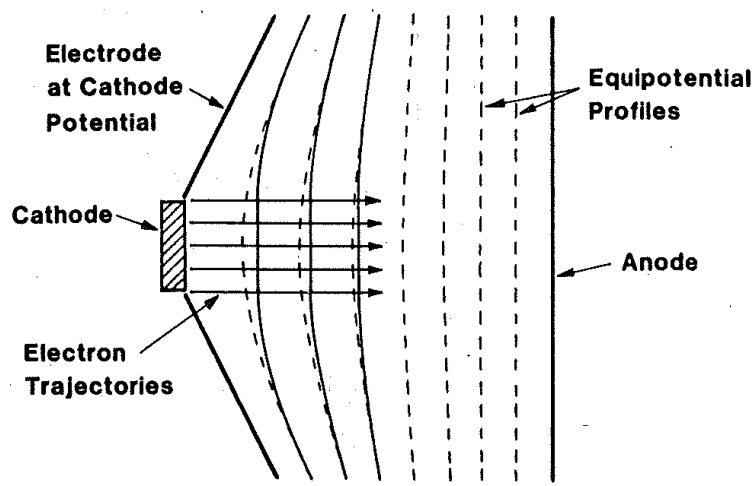
C=0 under the boundary conditions: ϕ=0 and $d\phi/dx$=0 at x=0; the condition $d\phi/dx$=~$x^{1/3}$=0 at x=0 implies that the special case ***electric field at cathode surface is null*** *(a steady state solution).*

$$\Rightarrow \frac{4}{3}\phi^{3/4} = 2\left(\frac{J}{\varepsilon_0}\right)^{1/2}\left(\frac{2e}{m}\right)^{-1/4} x \Rightarrow \phi(x) = V_0\left(\frac{x}{d}\right)^{4/3} \text{ with } J = \frac{4}{9}\varepsilon_0\left(\frac{2e}{m}\right)^{1/2}\frac{V_0^{3/2}}{d^2}$$

$$\boxed{J = 2.33\times10^{-6}\frac{V_0^{3/2}}{d^2}\left[A/m^2\right]}$$

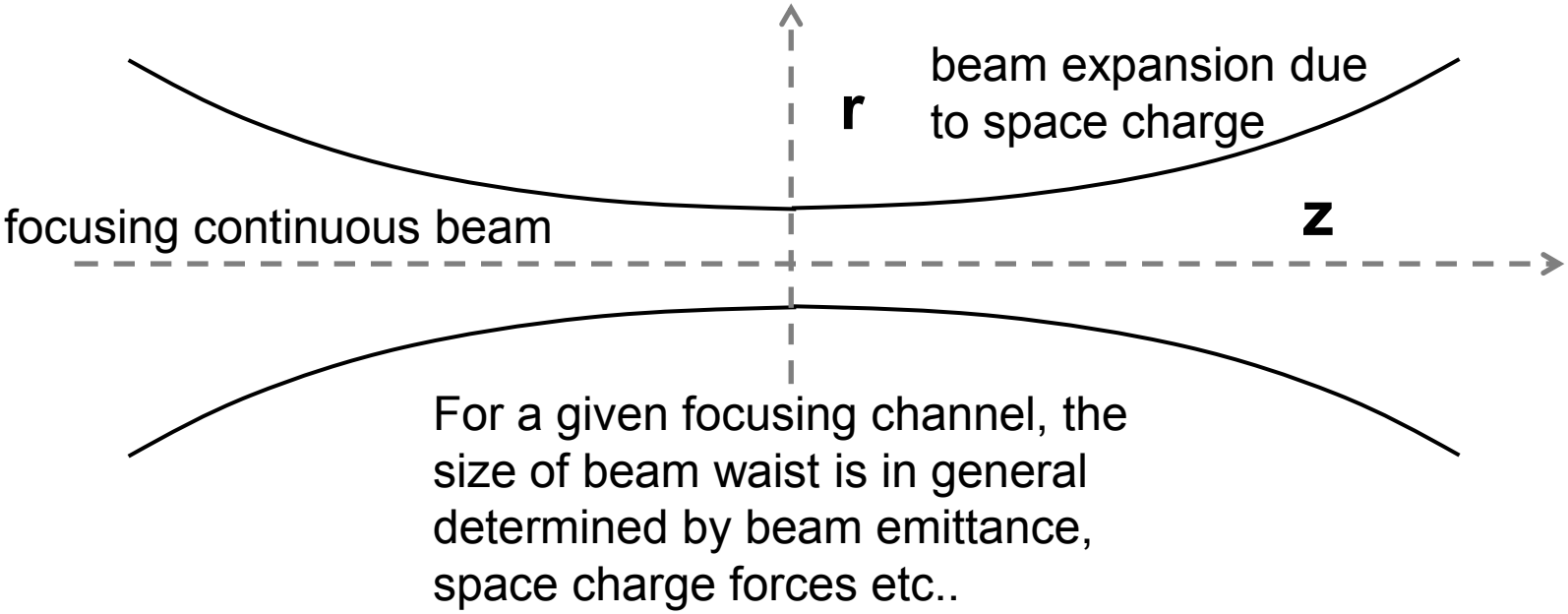
Child-Langmuir law

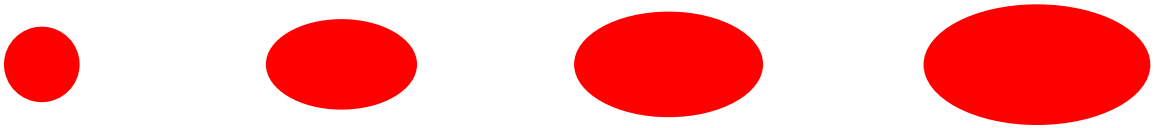




- Electrostatic repulsion forces between electrons tend to diverge the beam
- The current density required in the electron beam is normally far greater than the emission density of the cathode
- Optimum angle for parallel beam is referred to as Pierce electrodes
- Conical diode is needed for convergent flow
- Defocusing effect of anode aperture has to be considered

140 kV Electron Gun System for TLS

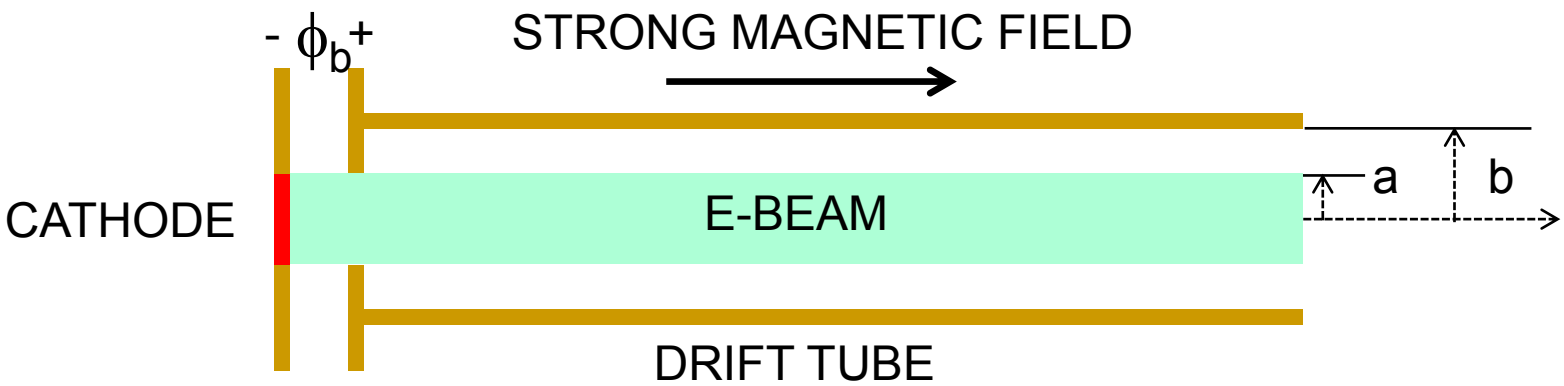


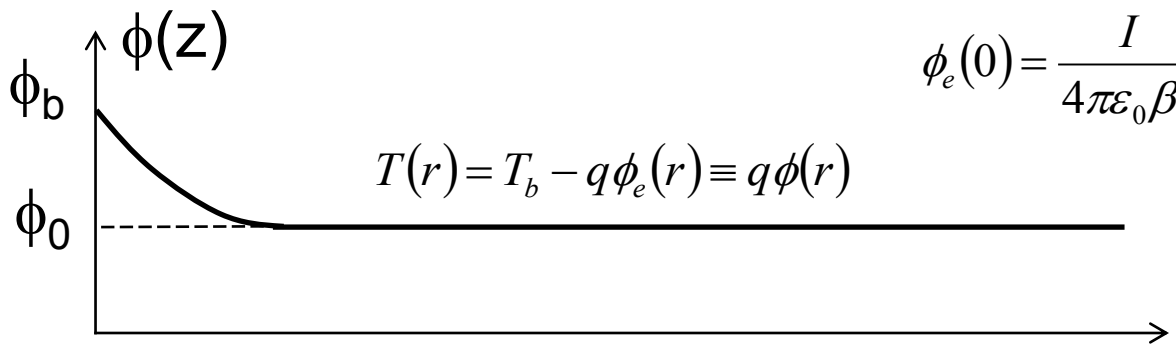


change of particle distribution in phase space due to space charge (transverse beam size, bunch length, divergence, energy spread etc..)

propagation of a continuous beam and a bunched beam in drift space

A Cylindrical Beam in a Strong Magnetic Field

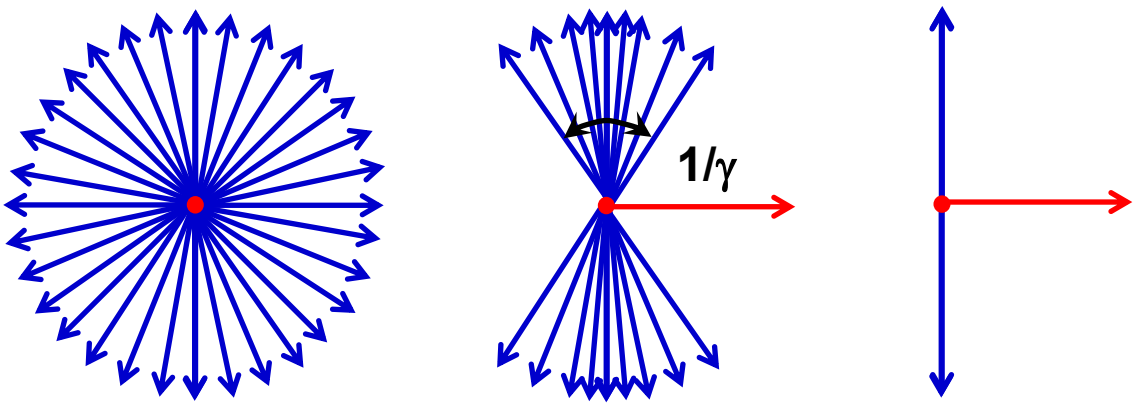




$$\phi_e(0) = \frac{I}{4\pi\varepsilon_0\beta c}\left(1 + 2\ln\frac{b}{a}\right)$$

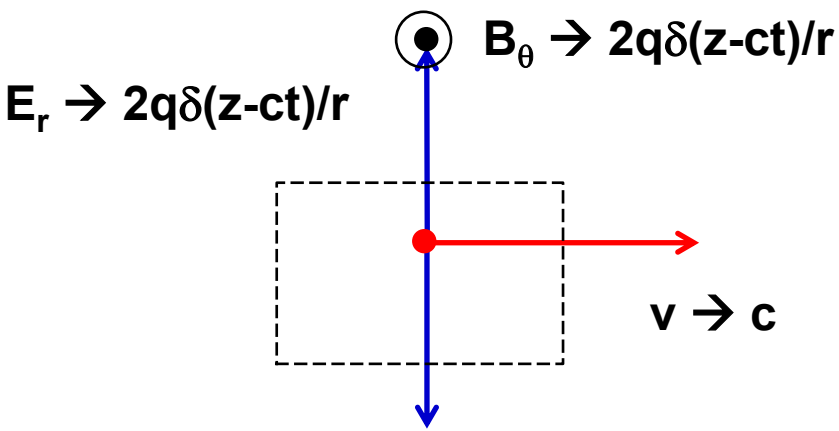
$$T(r) = T_b - q\phi_e(r) \equiv q\phi(r)$$

- Particles enter the drift tube with kinetic energy $q\phi_b$.
- A new potential will be setup in the drift tube which will reduce the K.E. of the particles according to energy conservation law.
- As the potential is strong enough, K.E. of particles completely convert into potential energy. There exists a beam current limit!!



v = 0 v < c v = c

electric fields of a uniformly moving charged particle



$$\vec{F} \approx -e\left(\vec{E} + \hat{z} \times \vec{B}\right) \rightarrow 0$$

Paraxial Approximation of Axisymmetric Electric and Magnetic Fields

- General assumptions:
  - Consider only *static* electric and magnetic fields at this point.
    That is, no time-varying fields and displacement currents are excluded.

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = 0 \qquad \nabla \times \vec{E} = 0$$

  - Electric and magnetic fields for axisymmetric lenses in a beam focusing system do not usually have azimuthal components (i.e. $E_\theta = 0; B_\theta = 0$).
  - Other field components have no azimuthal dependence

$$\frac{\partial E_z}{\partial \theta} = \frac{\partial E_r}{\partial \theta} = 0 \qquad \frac{\partial B_z}{\partial \theta} = \frac{\partial B_r}{\partial \theta} = 0$$

  - In paraxial approx., fields are calculated at small radii from the system axis with the assumption that the field vectors make small angles with the axis

$$E_r << E_z \qquad B_r << B_z$$

- The axial and radial fields in paraxial approximation are:

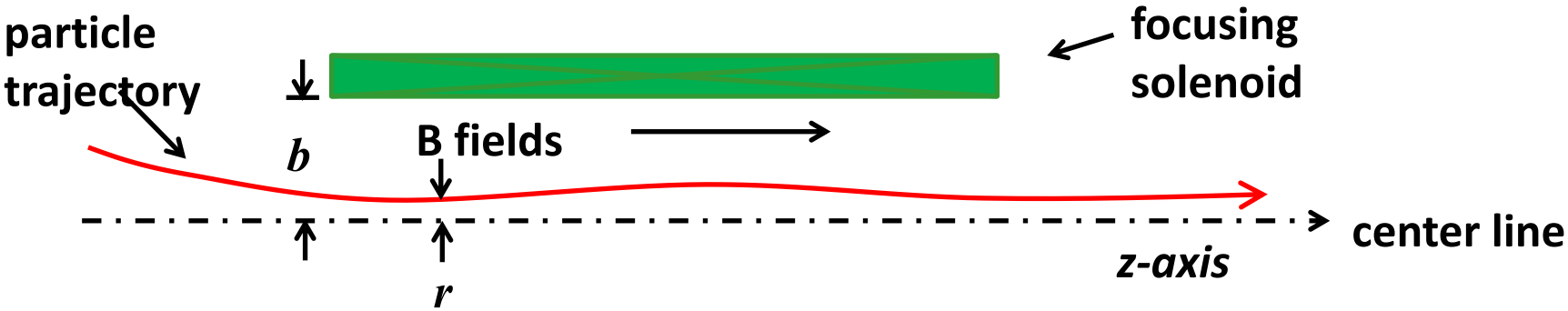
$$E_r(r,z) = -\frac{r}{2}\frac{\partial E}{\partial z} + \frac{r^3}{16}\frac{\partial^3 E}{\partial z^3} - \cdots \qquad (1)$$

$$E_z(r,z) = E - \frac{r^2}{4}\frac{\partial^2 E}{\partial z^2} + \frac{r^4}{64}\frac{\partial^4 E}{\partial z^4} - \cdots \qquad (2)$$

$$B_r(r,z) = -\frac{r}{2}\frac{\partial B}{\partial z} + \frac{r^3}{16}\frac{\partial^3 B}{\partial z^3} - \cdots \qquad (3)$$

$$B_z(r,z) = B - \frac{r^2}{4}\frac{\partial^2 B}{\partial z^2} + \frac{r^4}{64}\frac{\partial^4 B}{\partial z^4} - \cdots \qquad (4)$$

Particle Dynamics in an Axis-symmetric System



- Derive linear equations of particle motion in which only terms up to first order in r and $r'=dr/dz$ are considered
- Assumptions:
  - Cylindrical beam; self-fields are neglected
  - Particle trajectories remain close to the axis. That is, $r << b$. And b is the solenoid or radii of electrodes that produce the electric and magnetic fields. This also implies that the slopes of the particle trajectories remain small (i.e., $r' << 1$ or $\dot{r} << \dot{z}$).
  - Azimuthal velocity v_θ must remains very small compared to the axial velocity (i.e., $r\dot{\theta} << \dot{z}$). Thus, in this linear approximation, we have

$$\dot{z} = \left(v^2 - \dot{r}^2 - r^2\dot{\theta}^2\right)^{1/2} \approx v$$

To describe the motion of a charged particle that moves close to the beam axis, ‘paraxial ray equation’ can be derived

$$\boxed{r'' + \frac{\gamma'}{\gamma\beta^2} r' + \frac{\gamma''}{2\gamma\beta^2} r + \left(\frac{qB}{2mc\beta\gamma}\right)^2 r - \frac{p_\theta^2}{m^2c^2\gamma^2\beta^2 r^3} = 0}$$

This equation is not a linear differential equation and it is not very useful in practice. However, the contribution from the last term can be neglected for a beam without angular momentum or if the particle with constant angular momentum, we can choose a rotating reference frame in which the last term can be avoided. Especially when we are interested only on the evolution of beam envelop r_m along z

$$\boxed{r_m'' + \frac{\gamma'}{\gamma\beta^2} r_m' + \frac{\gamma''}{2\gamma\beta^2} r_m + \left(\frac{qB}{2mc\beta\gamma}\right)^2 r_m = 0}$$

For a simple beam moving at constant energy in a field free region, its envelop diverges at a constant rate along the beam axis z (i.e. r'_m = const.).

Solenoid Magnetic Lenses

Recall paraxial ray equation:

$$r'' + g_1(z)r' + g_2(z)r = 0$$

With

$$g_1(z) = \frac{\gamma'}{\beta^2\gamma} \qquad g_2(z) = \frac{\gamma''}{2\beta^2\gamma} + \frac{\omega_L^2}{\beta^2 c^2}$$

For pure magnetic field, particle energy conserved. Therefore, $\gamma' = \gamma'' = 0$

$$r'' + K^2 r = 0$$

With

$$K^2 = \left(\frac{qB}{2mc\gamma\beta}\right)^2 = \frac{\omega_L^2}{\beta^2 c^2}$$

$$\Rightarrow \qquad r_2' - r_1' = -\int_{z_1}^{z_2} K^2 r dz$$

Since K^2 is always positive and if the particle does not cross the axis inside the lens field (i.e., r > 0), the lens is focusing.

For thin lens, r can be treated as a constant and taken out from the integral

$$r' = -r\left(\frac{q}{2mc\beta\gamma}\right)^2 \int_{z_1}^{z_2} B^2 dz$$

Further, if $f_2 = f_1$, then

$$\frac{1}{f} = -\frac{r'}{r} = \left(\frac{q}{2mc\beta\gamma}\right)^2 \int_{z_1}^{z_2} B^2 dz$$

On the other hand, the particle is rotated by an angle $\theta_r = -\int_{z_1}^{z_2} K dz$

But for solenoids, integration of H field along z equals to NI (number of ampere turns of the coil)

$$\theta_r = -\mu_0 \frac{q}{2mc\beta\gamma} NI$$

Hard edge approximation:

1. The field in the region 0 < z < l is assumed to be uniform and zero outside this region, where l is the effective length of the solenoid
2. D/l << 1

From energy conservation law,

$$lB_0^2 = \int_{-\infty}^{\infty} B^2(z)dz$$

$$\Rightarrow \quad l = \frac{1}{B_0^2}\int_{-\infty}^{\infty} B^2(z)dz \qquad (44)$$

effective length

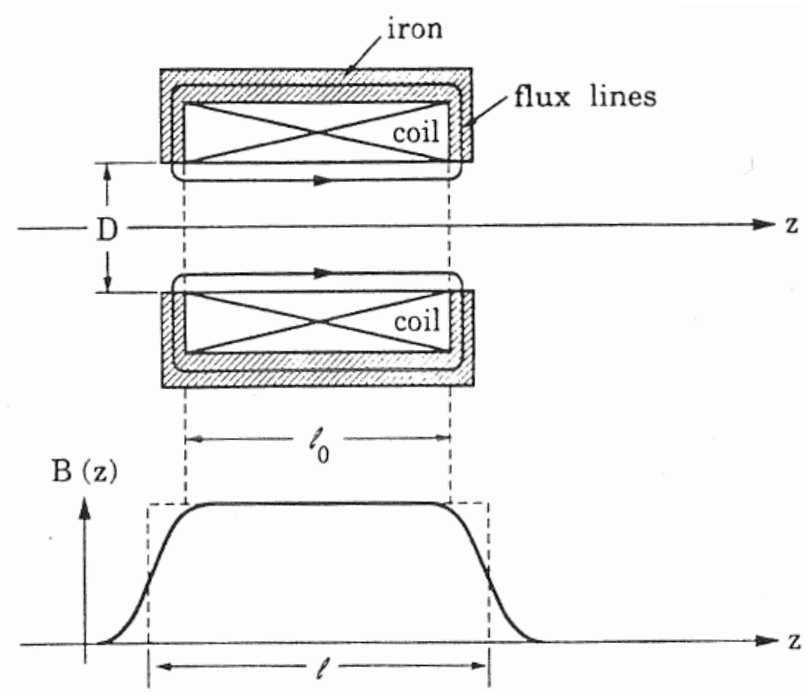


Figure 3.10. Solenoid lens with iron shield.

Initial conditions at z = 0, (i.e. $r = r_0, r' = 0$)

$$\begin{cases} r = r_0 \cos Kz \\ r' = -r_0 K \sin Kz \end{cases} \qquad (45)$$

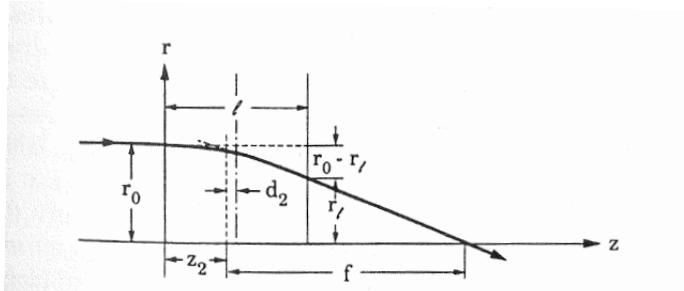


Figure 3.11. Trajectory entering solenoidal field with zero slope.

At the end of the solenoid,

$$\begin{cases} r_l = r_0 \cos \phi \\ r_l' = \dfrac{r_0}{l} \phi \sin \phi \end{cases}$$

Image rotation:

$$\theta_r = -\int_0^l K dz = -Kl = -\phi$$

where

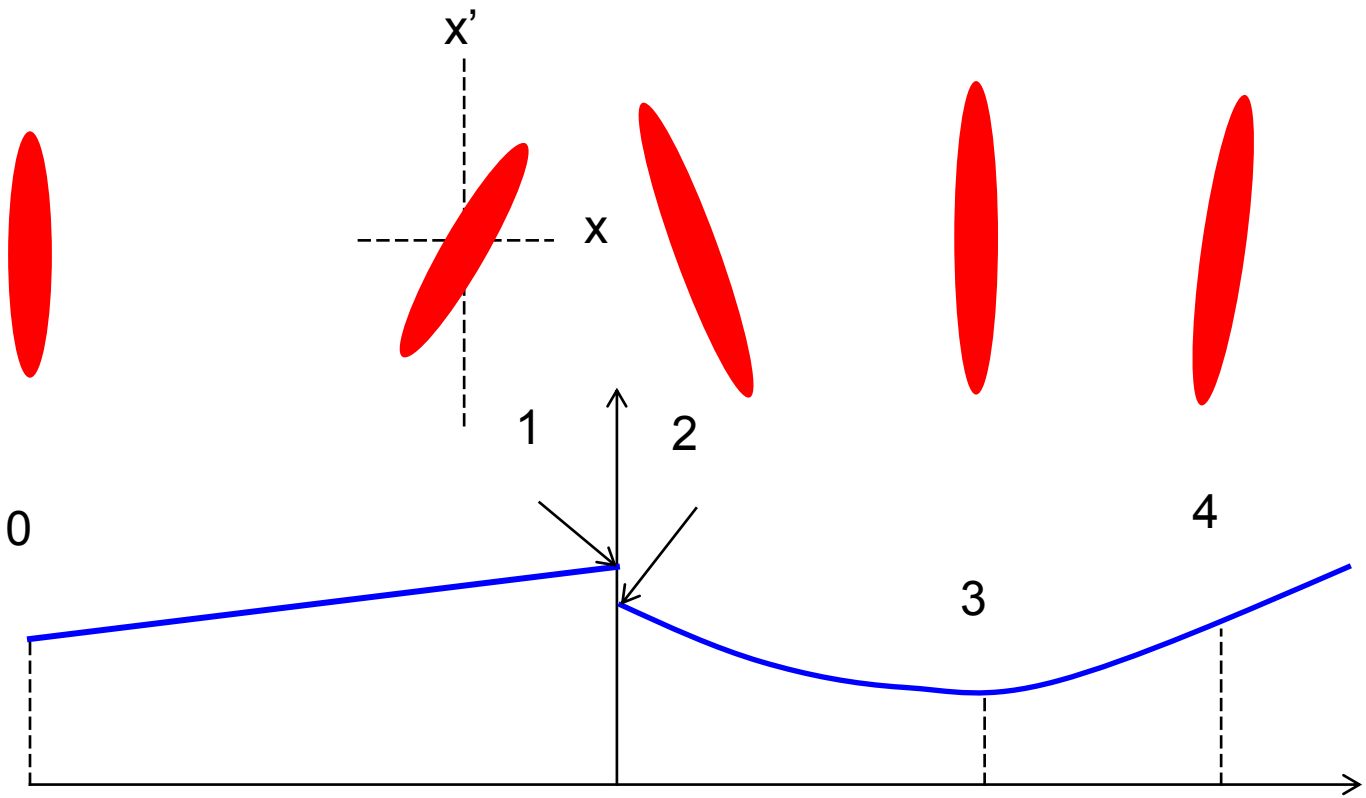
$$\phi = Kl = \frac{qB_0 l}{2mc\beta\gamma} = \frac{\mu_0 qNI}{2mc\beta\gamma} \qquad (46)$$

The focal length

$$\frac{1}{f} = -\frac{r_l'}{r_0} = \frac{\phi \sin \phi}{l} \approx \frac{\phi^2}{l} = K^2 l \qquad (47)$$

thin lens approximation

Effects of a Lens on Trace Space Ellipse and Beam Envelope



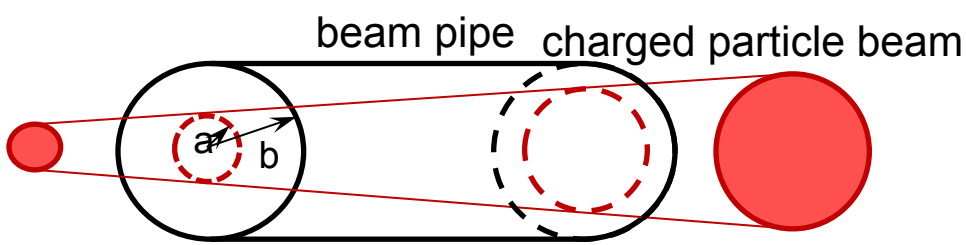
Uniform Beam Model

- The beam is assumed to have a sharp boundary

$$\left.\begin{aligned}\rho &= const\\ J &= const\end{aligned}\right\} \quad \text{inside the boundary}$$

$$\rho = J = 0 \qquad \text{everywhere outside the boundary}$$

- The uniformity of charge and current densities assures that the transverse electric and magnetic self-fields and the associated forces are linear functions of transverse coordinates (see below)
- This beam model allows us to extend the linear beam optics to include the space charge forces.



evolution of beam envelop along the propagation direction is exaggerated!!

- A axisymmetric *laminar* beam
- Particle trajectories obey paraxial assumption that the angle with the z-axis is small.
- The variation of beam radius along z-axis is slow enough that E_z and B_r can be neglected.

Based on the above assumptions, we have J, ρ and $v_z \approx v$ are all constant values across the beam.

Therefore, with $\rho_0 = I / a^2 \pi v$ denoting the charge density of the beam, we obtain

$$\left.\begin{aligned} J_z &= J = \frac{I}{a^2 \pi} \\ \rho &= \rho_0 = \frac{I}{a^2 \pi v} \qquad \text{for } 0 \le r \le a \\ \rho &= J = 0 \qquad \text{for } r > a \end{aligned}\right\} \quad (1)$$

Since we assumed the electric field has only a radial component and by Gauss' law,

$$E_r = \frac{\rho r}{2\varepsilon_0} = \frac{Ir}{2\pi\varepsilon_0 a^2 v} \qquad \text{for } 0 \le r \le a \qquad (2a)$$

$$E_r = \frac{I}{2\pi\varepsilon_0 v r} \qquad \text{for } r > a \qquad (2b)$$

the Lorentz force exerted on an electron in the beam has radial component only and is a linear function of r.

The magnetic field, on the other hand has only an azimuthal component, is obtained by applying Ampere's circuital law:

$$\left.\begin{aligned} B_\theta &= \mu_0 \frac{Ir}{2\pi a^2} && \text{for } 0 \le r \le a \\ B_\theta &= \mu_0 \frac{I}{2\pi r} && \text{for } r > a \end{aligned}\right\} \quad (4)$$

Integrate the electrostatic field along an arbitrary path from r = a to r = b,

$$\left.\begin{aligned} \phi(r) &= V_s\left(1 + 2\ln\left(\frac{b}{a}\right) - \frac{r^2}{a^2}\right) && \text{for } 0 \le r \le a \\ \phi(r) &= 2V_s \ln\left(\frac{b}{r}\right) && \text{for } r > a \end{aligned}\right\} \quad (5)$$

by setting that $\phi = 0$ at r = b, where

$$V_s = \frac{\rho_0 a^2}{4\varepsilon_0} = \frac{I}{4\pi\varepsilon_0\beta\gamma} \approx \frac{30I}{\beta} \quad (6)$$

The peak potential on the beam axis (r = 0) is thus $\phi(0) = V_0 = V_s[1 + 2\ln(b/a)]$. And the maximum electric field at the beam edge is $E_a = \frac{2V}{a} \approx 60I/\beta a$

The motion of a beam particle in such field is described by the radial force equation

$$\frac{d}{dt}\left(\gamma m\dot{r}\right) = \gamma m\ddot{r} = q\left(E_r - v_z B_\theta\right) \tag{8}$$

where we dropped the force term $qr\dot{\theta}B_z$ that $r\dot{\theta}$ is negligibly small and $\gamma = const.$ because there is no external acceleration.

Substitution for E_r from the first equation of (2), B_θ from the first equation of (3) and with $\varepsilon_0\mu_0 = c^{-2}$, $\dot{z} = v = \beta c$, we have

$$\gamma m\ddot{r} = \frac{qIr}{2\pi\varepsilon_0 a^2 \beta c}\left(1-\beta^2\right) \tag{9}$$

with $\ddot{r} = v_z^2 \dfrac{d^2 r}{dz^2} = \beta^2 c^2 r''$ we have

$$r'' = \frac{qIr}{2\pi\varepsilon_0 a^2 mc^3 \beta^3 \gamma^3} \tag{10}$$

Define characteristic current (Alfve'n current) I_0 as

$$I_0 = \frac{4\pi\varepsilon_0 mc^3}{q} \approx \frac{1}{30}\frac{mc^2}{q} \tag{11}$$

which is approx. 17 kA for electrons and 31 (*A*/*Z*) MA for ions of mass number A and charge number Z.

Define “generalized perveance” *K* such that

$$K = \frac{I}{I_0}\frac{2}{\beta^3\gamma^3} \tag{12}$$

In terms of generalized perveance, the equation of motion can be expressed as

$$r'' = \frac{K}{a^2} r \tag{13}$$

Under the condition of laminar flow, the trajectories of all particles are similar and scale with the factor r/a. That is, the particle at r=a will always remain at the boundary of the beam. Thus, by setting r = a = r_m, we obtain

$$r_m r_m'' = K \tag{14}$$

From the paraxial ray equation and considering the effects of finite emittance and linear space charge, we obtained the beam envelop equation:

$$\boxed{r_m'' + \frac{\gamma'}{\gamma\beta^2} r_m' + \frac{\gamma''}{2\gamma\beta^2} r_m + \left(\frac{qB}{2mc\beta\gamma}\right)^2 r_m - \frac{p_\theta^2}{m^2c^2\gamma^2\beta^2 r_m{}^3} - \frac{\varepsilon_n^2}{\beta^2\gamma^3 r_m^3} - \frac{K}{r_m} = 0} \tag{15}$$

Electron acceleration in a photo-cathode rf gun

Electric field along the axis

$$E_z = E_0 \cos kz \sin(\omega t + \phi_0) \quad (1)$$

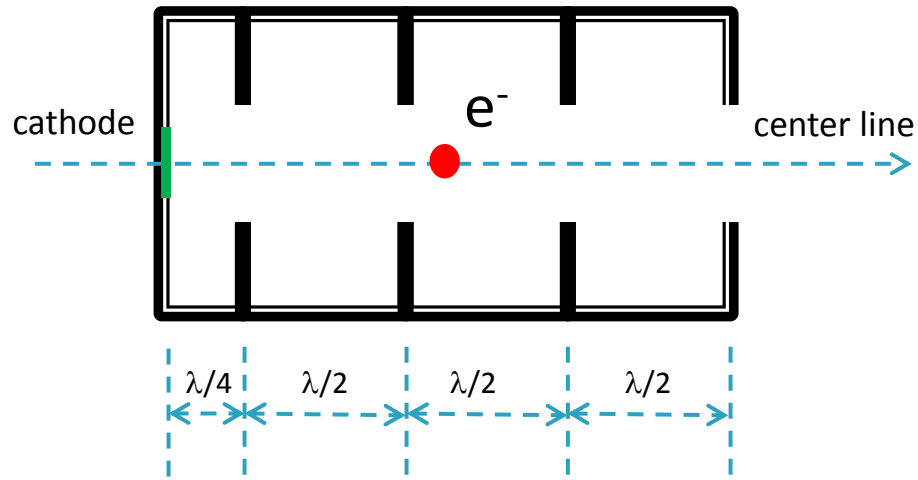
define electron phase with respect to rf field as

$$\phi = \omega t - kz + \phi_0 \quad (2)$$

then $\phi = k\int_0^z \left(\frac{dz}{v} - dz \right) + \phi_0$

or $\phi = k\int_0^z \left(\frac{\gamma}{\sqrt{\gamma^2 - 1}} \right) dz + \phi_0 \quad (3)$

acceleration of an electron in the gun is



- The electron emitter is laser driven photo-cathode
- Operating mode in each cell is TM_{010}–like. Coupled cavity mode is π-mode.
- The cavity cells are on-axis coupled through apertures
- Lengths of cells are about $\lambda/2$
- The first cell is shorter because electrons move slower there.

approximation near the cathode

$$\frac{d\gamma}{dz} \approx \frac{eE_0}{mc^2}\sin\phi_0 \qquad (5)$$

integrate this equation along the z-axis

$$\tilde{\gamma} = 1 + 2\alpha\sin\phi_0 \cdot kz \qquad (6)$$

where $\alpha = eE_0/2mc^2k$ is a dimensionless parameter. With this approx. of $d\gamma/dz$,

$$\phi = \frac{1}{2\alpha\sin\phi_0}\left[\sqrt{\tilde{\gamma}^2 - 1} - (\tilde{\gamma} - 1)\right] + \phi_0 \qquad (7)$$

Substitute (7) back to (4) and integrate, we have

$$\gamma = 1 + \alpha\left[kz\sin\phi + \frac{1}{2}\cos\phi - \cos(\phi + 2kz)\right] \qquad (8)$$

For $\tilde{\gamma}$ >>1, the asymptotic value of ϕ is

$$\phi \to \phi_\infty = \frac{1}{2\alpha\sin\phi_0} + \phi_0 \qquad (9)$$

analysis of electron dynamics suggested that transverse emittance is minimized when $\phi_\infty = \pi/2$

Therefore, initial phase should be chosen such that

$$\left(\frac{\pi}{2} - \phi_0\right)\sin\phi_0 = \frac{1}{2\alpha} \qquad (10)$$

For initial phase spread of electron $\Delta\phi_0$, spread of electron phase after acceleration is

$$\Delta\phi_\infty = \left(1 - \frac{\cos\phi_0}{2\alpha\sin^2\phi_0}\right)\cdot\Delta\phi_0 \qquad (11)$$

the bunch length dz = -$\Delta\phi$/k will in general be compressed during acceleration.

The rf effects on *electron distribution in longitudinal phase space* can be derived from the approx. expression of γ at the end of the n+1/2 cavity:

$$\gamma = 1 + \alpha\left[\left(n + \tfrac{1}{2}\right)\pi\sin\phi + \cos\phi\right] \qquad (12)$$

If we write

$$p_z = \langle p_z \rangle + \Delta p_z$$
$$z = \langle z \rangle + \Delta z \qquad (13)$$

then the longitudinal emittance by definition becomes (14)

$$\varepsilon_z = \sqrt{\langle(\Delta p_z)^2\rangle\langle(\Delta z)^2\rangle - \langle\Delta p_z\rangle^2\langle\Delta z\rangle^2}$$

(15)

in terms of $\Delta\phi$ and $\Delta\gamma$

$$\varepsilon_z = \frac{1}{k}\sqrt{\langle(\Delta\gamma)^2\rangle\langle(\Delta\phi)^2\rangle - \langle\Delta\gamma\rangle^2\langle\Delta\phi\rangle^2}$$

at the gun exit

$$\langle\gamma\rangle + \Delta\gamma = 1 + \alpha\left[\left(n+\tfrac{1}{2}\right)\pi\sin\left(\langle\phi\rangle + \Delta\phi\right) + \cos\left(\langle\phi\rangle + \Delta\phi\right)\right] \tag{16}$$

setting $\langle\phi\rangle = \pi/2$ to minimize transverse emittance

$$\Delta\gamma = \alpha\cos\Delta\phi - 1 - \sin\Delta\phi \tag{17}$$

expand $\Delta\gamma$ for small $\Delta\phi$,

$$\Delta\gamma = -\alpha\Delta\phi - \frac{1}{2}\left(\gamma_f - 1\right)\left(\Delta\phi\right)^2 + \frac{\alpha}{3!}\left(\Delta\phi\right)^3 + \ldots \tag{18}$$

we obtain,

$$\varepsilon_z^{rf} = \frac{1}{k}\left(\gamma_f - 1\right)\sqrt{\left\langle\left(\Delta\phi\right)^4\right\rangle\left\langle\left(\Delta\phi\right)^2\right\rangle} \tag{19}$$

for Gaussian distribution,

$$\varepsilon_z^{rf} = \sqrt{3}\left(\gamma_f - 1\right)k^2\sigma_z^3 \tag{20}$$

Longitudinal emittance scales as the third power of bunch length and as the square of rf frequency.

Assuming the following longitudinal E-field,

$$E_z = E(z)\cos kz \sin(\omega t + \phi_0)$$

E_r, B_θ can be found from Maxwell's equations, that is

$$E_r = -\frac{r}{2}\frac{\partial E_z}{\partial z} \qquad cB_\theta = \frac{r}{2c}\frac{\partial E_z}{\partial t}$$

The radial force is given by

$$F_r = e(E_r - \beta c B_\theta)$$

then the equation describing radial motion is

$$\frac{dp_r}{dt} = \frac{1}{mc}F_r$$

with

$$F_r = er\left\{-\frac{1}{2c}\frac{d}{dt}\left[E(z)\sin kz\cos(\omega t + \phi_0)\right] - \frac{1}{2}\left[\frac{dE(z)}{dz}\right]\cos kz \sin(\omega t + \phi_0) + \frac{\beta}{2}\left[\frac{dE(z)}{dz}\right]\sin kz \cos(\omega t + \phi_0)\right\}$$

Assume the transverse deflection is small, r can be regarded as constant, integrate the equation of radial motion gives the radial impulse

$$p_r = \frac{1}{mc}\int_0^f F_r dt$$

$$\Rightarrow \quad p_r = p_{r0} + \alpha k r\left[\beta \cos kz_f \sin(\omega t + \phi_0) - \sin kz_f \cos(\omega t + \phi_0)\right]$$

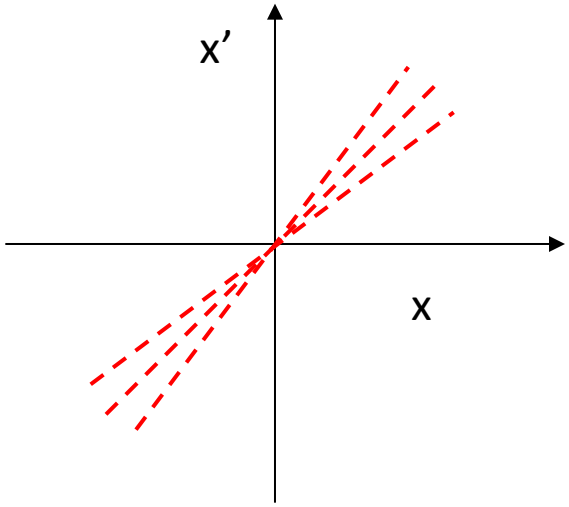
and assume $p_r = 0$ at $t = 0$; $v \rightarrow c$ near cavity exit

$$p_r = \alpha k r \sin\phi$$

here ϕ is the rf phase at the cavity exit

in Cartesian coordinates,

$$p_x = \beta\gamma x' = (\alpha k \sin\phi)x$$



Electron trajectories in trace-space is therefore are lines with different slopes Correspond to different ϕ

The normalized transverse emittance is therefore

$$\varepsilon_x^{rf} = \sqrt{\left\langle p_x^2 \right\rangle \left\langle x^2 \right\rangle - \left\langle p_x x \right\rangle^2} = \alpha k \left\langle x^2 \right\rangle \sqrt{\left\langle \sin^2 \phi \right\rangle - \left\langle \sin \phi \right\rangle^2}$$

and therefore

$$\varepsilon_x^{rf} = \alpha k \left\langle x^2 \right\rangle \sqrt{\left\{ \left[\left\langle (\Delta\phi)^2 \right\rangle^2 - \frac{1}{3} \left\langle (\Delta\phi)^4 \right\rangle \right] \cos^2 \left\langle \phi \right\rangle + \frac{1}{4} \left[\left\langle (\Delta\phi)^4 \right\rangle - \left\langle (\Delta\phi)^2 \right\rangle^2 \right] \sin^2 \left\langle \phi \right\rangle \right\}}$$

the emittance is minimized when $\langle\phi\rangle = 90°$ such that

$$\varepsilon_x^{rf} = \alpha k \frac{\left\langle x^2 \right\rangle}{2} \sqrt{\left\{ \left[\left\langle (\Delta\phi)^4 \right\rangle - \left\langle (\Delta\phi)^2 \right\rangle^2 \right] \right\}}$$

for Gaussian distribution,

$$\varepsilon_x^{rf} = \frac{\alpha k \left\langle x^2 \right\rangle \sigma_\phi^2}{\sqrt{2}}$$

Improvement of Beam Brightness by RF Bunch Compression

The equation of motion of a particle accelerated by a traveling-wave

$$\frac{d}{dt} mc\beta\gamma = mc\gamma^3 \frac{d\beta}{d\phi}\frac{d\phi}{dt} = qE_0 \cos\phi(z,t)$$

where the phase of the traveling-wave is

$$\phi(z,t) = \omega t - \frac{2\pi z}{\lambda}$$

the phase motion is

$$\frac{d\phi}{dt} = \frac{2\pi c}{\lambda}(1-\beta)$$

Since $\beta<1$, ϕ increases with time. The particle falls behind the initial phase on the wave

expressing γ in the equation of motion in β, we have

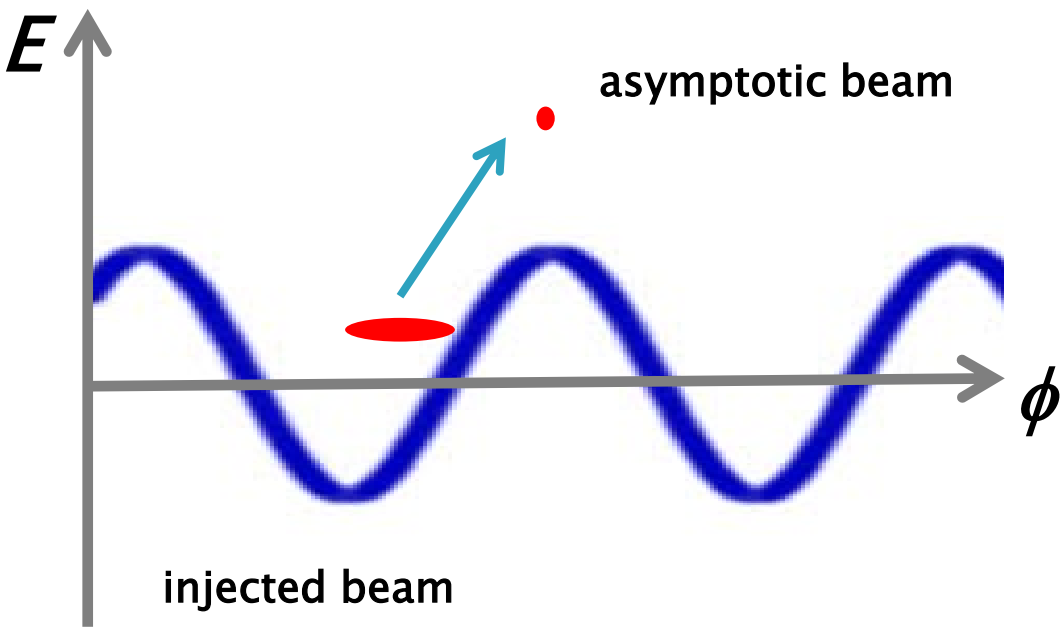
$$\frac{1}{(1+\beta)\sqrt{1-\beta^2}}\frac{d\beta}{d\phi} = \frac{qE_0\lambda}{2\pi mc^2}\cos\phi \quad \xrightarrow{\text{integration}} \quad \sin\phi = \sin\phi_i + \frac{2\pi mc^2}{qE_0\lambda}\left(\sqrt{\frac{1-\beta_i}{1+\beta_i}} - \sqrt{\frac{1-\beta}{1+\beta}}\right)$$

at asymptotic phase ($\beta \rightarrow 1$),

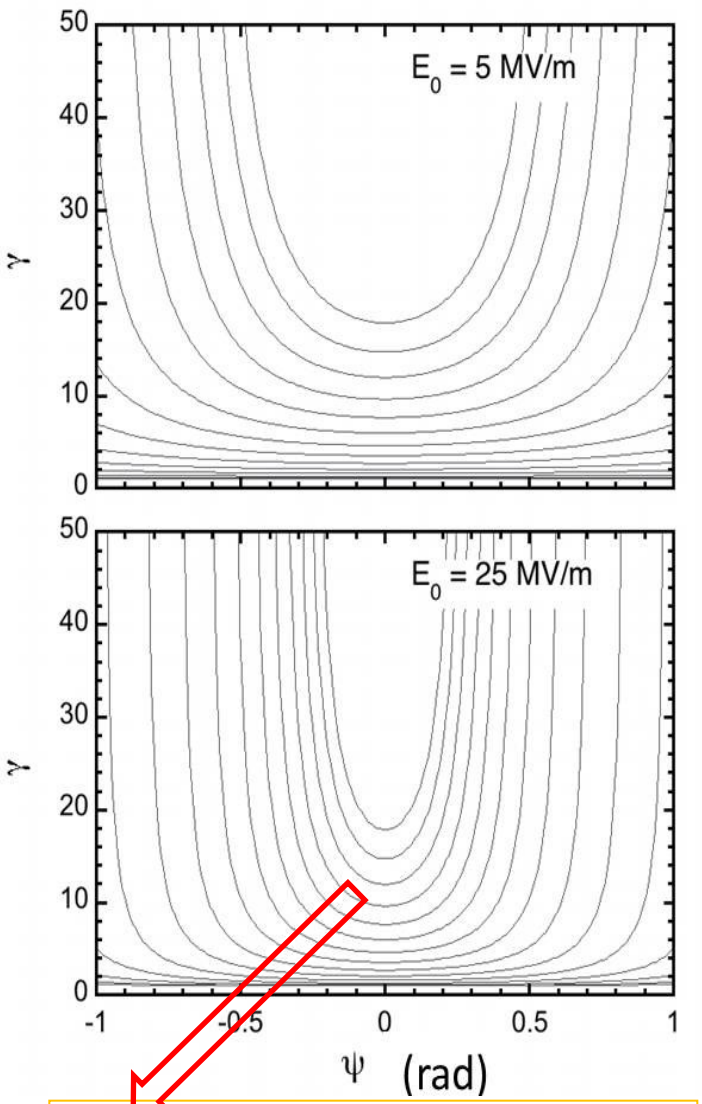
$$\sin\phi_\infty = \sin\phi_i + \frac{2\pi mc^2}{qE_0\lambda}\sqrt{\frac{1-\beta_i}{1+\beta_i}}$$

if we want to set ϕ_∞ at the rf crest for efficient acceleration, the condition is

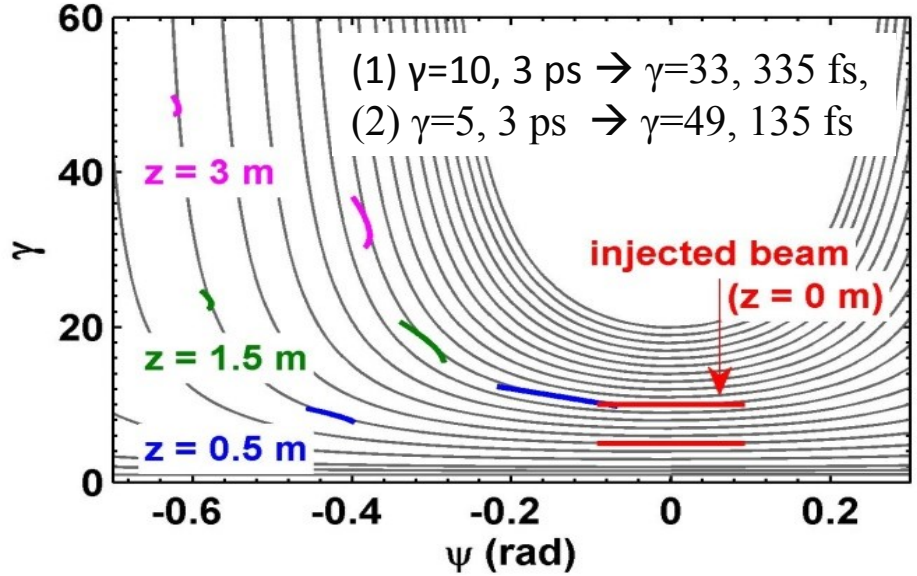
$$\sin\phi_i = -\frac{2\pi mc^2}{qE_0\lambda}\sqrt{\frac{1-\beta_i}{1+\beta_i}}$$



RF Bunch Compression in an RF Linac



nonlinearity of the equi-potential line at higher injection energy is stronger



- Injected bunch: all of the electrons distributed on a single equi-potential line → the ideal case
- rf nonlinearity leads to asymmetric distribution of longitudinal phase space → the limitation of compression in bunch length
- Solution → injection of lower energy beam !!
  - flat region covered widely
  - Benefit to relieve this limitation

Generally, thermionic rf gun ~ 2 MeV ;

photocathode gun ~ 4 MeV

→ Thermionic rf gun injector is suitable with velocity bunching method for bunch