# Fundamentals of Free-Electron Laser

**Yen-Chieh Huang 黃衍介**
ychuang@ee.nthu.edu.tw, tel: 886-3-5162340, fax: 886-3-5162330

清華大學電機工程學系/光電研究所/物理系
Department of Electrical Engineering/Institute of Photonics Technologies/
Department of Physics
National Tsinghua University, Hsinchu, Taiwan

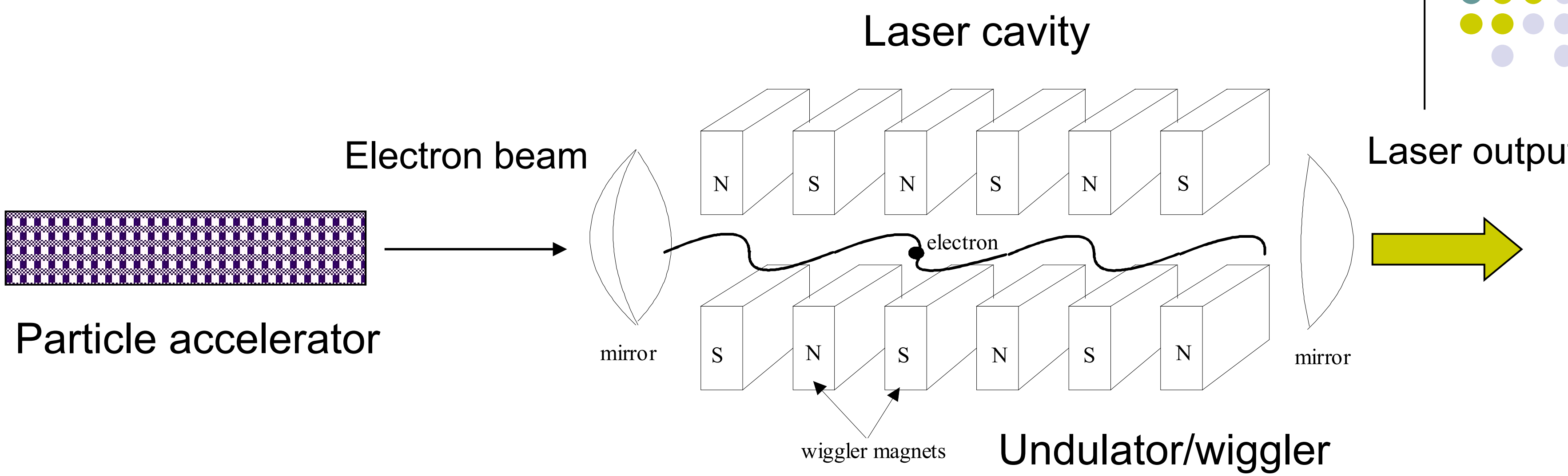
# Outlines

1. Spontaneous emission – Compton scattering/Thompson scattering/undulator radiation

2. Stimulated emission – wave/particle energy exchange $\rightarrow$ laser gain

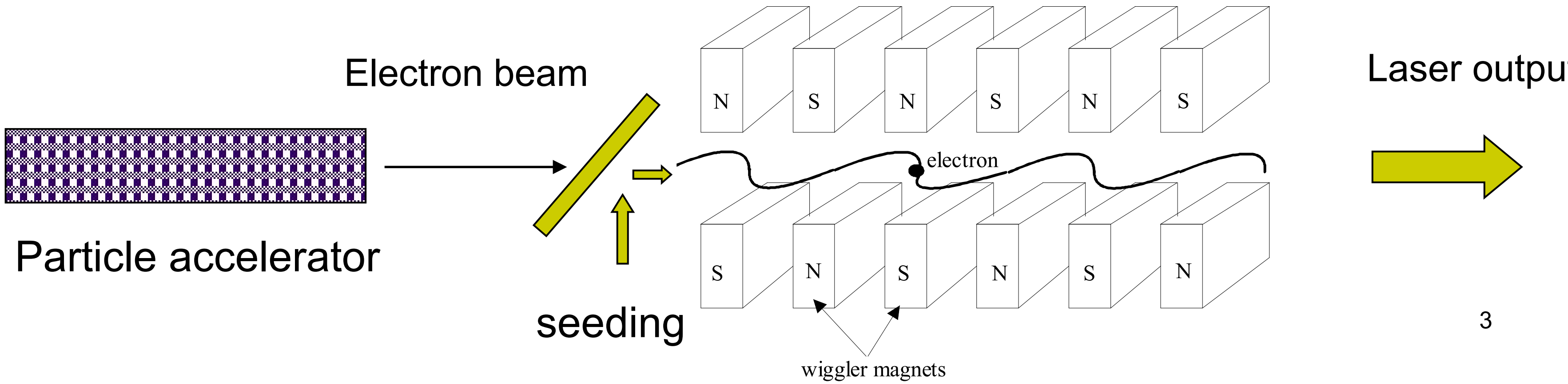
3. Requirements for FEL Oscillator: buildup time, energy spread, emittance, saturation power, etc.

# Low gain → free-electron Laser Oscillator

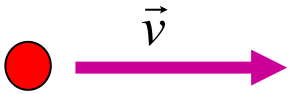



# High gain → free-electron Laser Amplifier

Self-amplified Spontaneous emission (SASE) FEL



# Parameters in Relativistic Mechanics




Moving particle

Lorentz factor $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$

where $\beta \equiv v/c$ , with $c$ = speed of light in vacuum.

Electron mass $m = \gamma m_0$ , $m_0$ = electron rest mass

Electron momentum: $p = mv = \gamma m_0 v$

Total electron energy: $\gamma m_0 c^2 = \sqrt{m_0^2 c^4 + p^2 c^2}$, $m_0 c^2$ = electron rest energy ~ 0.5 MeV

In laboratory frame: length $L$

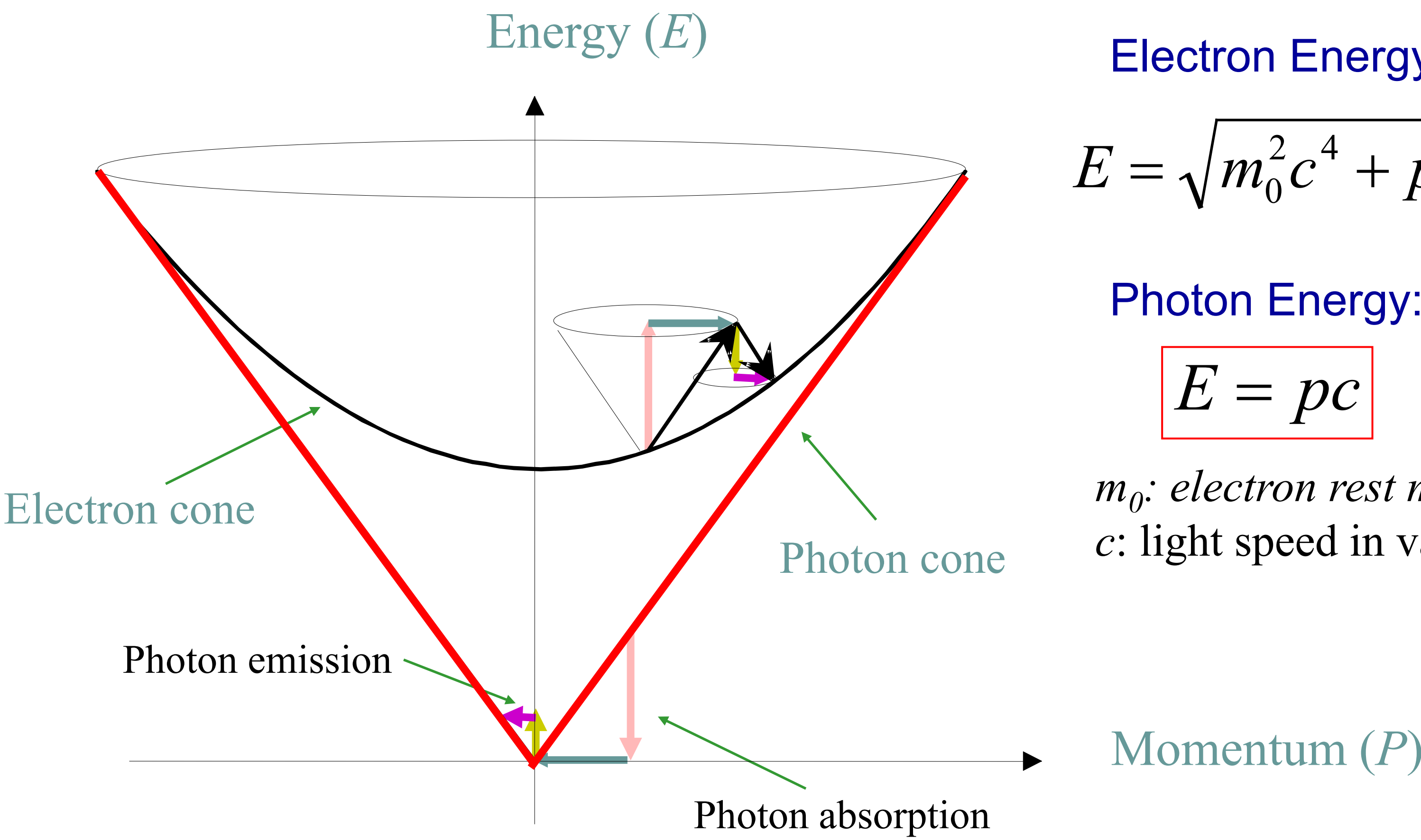
In electron frame: length $L/\gamma \leftarrow$ Lorentz contraction

In the relativistic regime $\beta \equiv v/c \sim< 1$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} >> 1 \Rightarrow \frac{1}{\beta} \sim 1 + \frac{1}{2\gamma^2} \Rightarrow \beta \sim 1 - \frac{1}{2\gamma^2}$$

# Photon-electron Energy Exchange in Free Space

requirements: energy conversation & momentum conservation




Electron Energy:

$$E = \sqrt{m_0^2 c^4 + p^2 c^2}$$

Photon Energy:

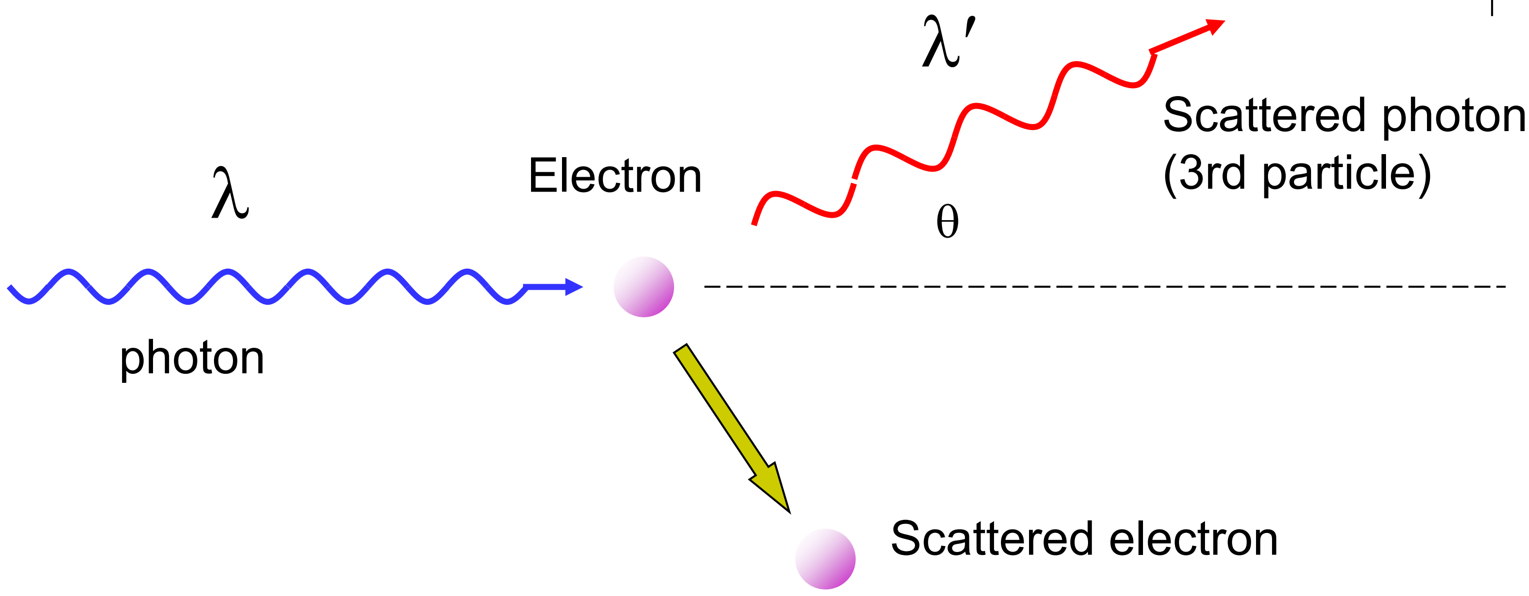
$$\boxed{E = pc}$$

*$m_0$: electron rest mass*
$c$: light speed in vacuum

**Energy-momentum diagram of Compton Scattering**

Photon-electron energy exchange is prohibited in a vacuum unless a third particle exists or is created

# Compton Scattering

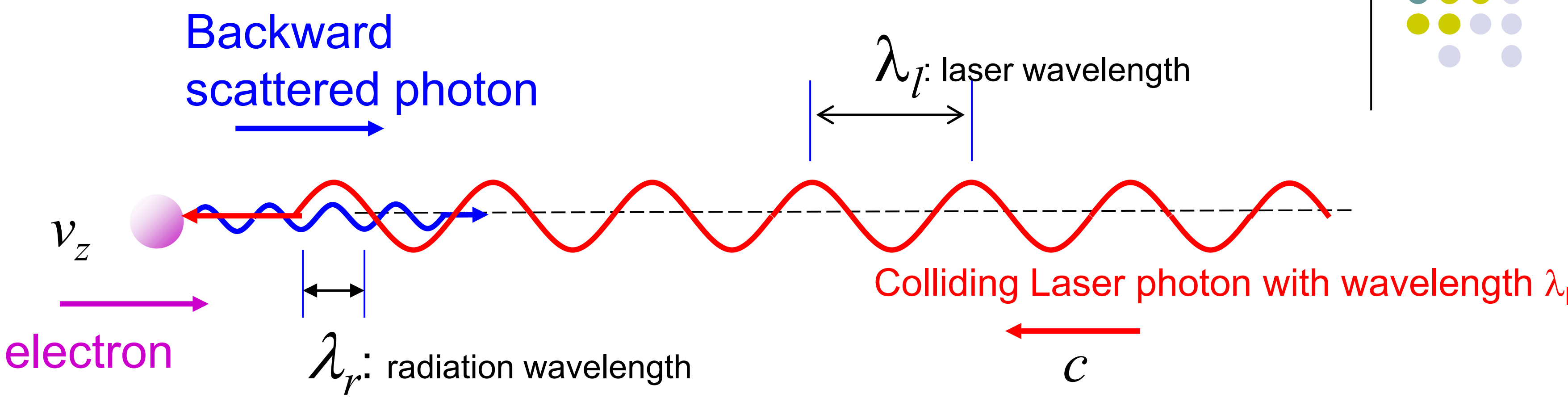



*λ: wavelength*
*h: Planck's constant*
*$m_0$: electron rest mass*
*c: vacuum wave speed*
*p*: momentum

Compton Effect $$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta)$$

# Thomson Back Scattering:

**Compton scattering with electron energy loss much less than the photon energy**




Double Doppler shift $f_r = f_l \underbrace{\sqrt{\frac{1+\beta_z}{1-\beta_z}}} \cdot \sqrt{\frac{1+\beta_z}{1-\beta_z}} = f_e' \times \underbrace{\sqrt{\frac{1+\beta_z}{1-\beta_z}}}_{\text{Doppler shift seen by a lab observer}} \Rightarrow \lambda_r = \frac{\lambda_l}{4\gamma_z^2}$

Radiation frequency in the lab

$f_e'$ : Doppler shifted laser frequency seen by electron

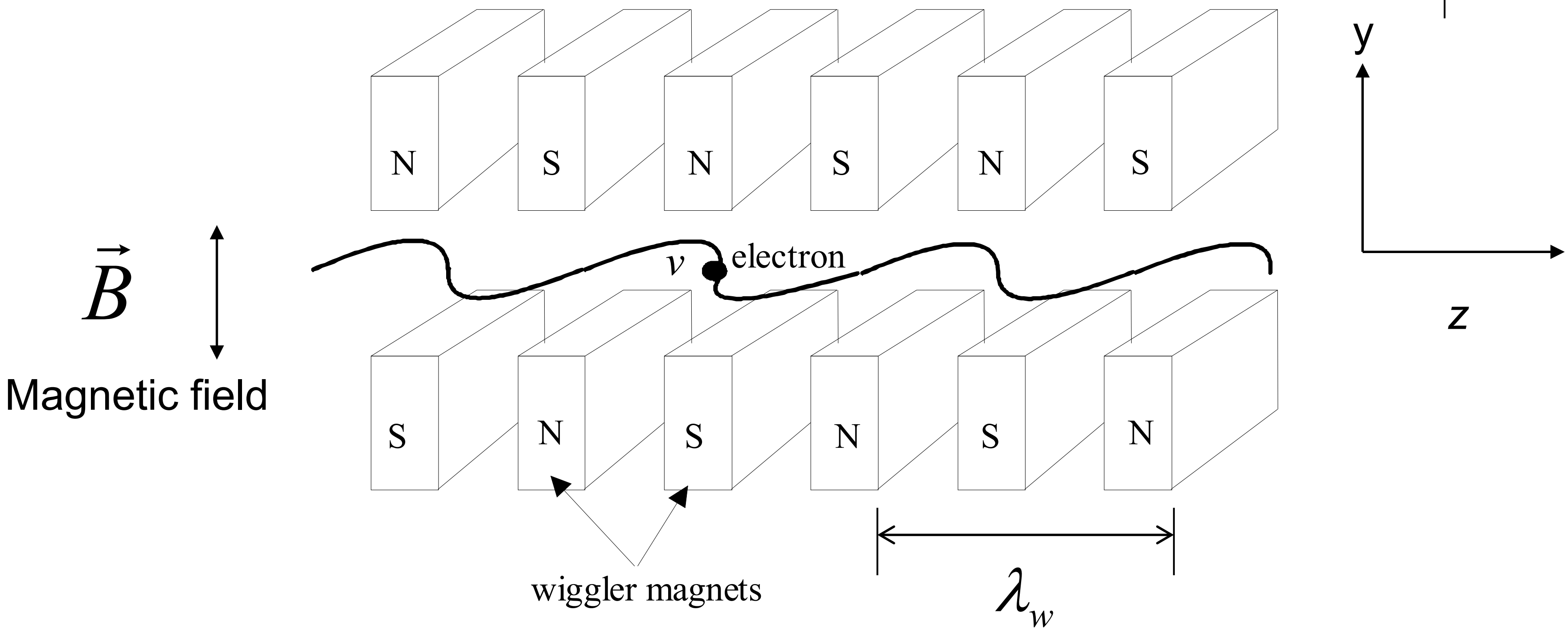
Given λ = 800 nm (Ti:sapphire laser), $\gamma \sim \gamma_z = 45$ (23 MeV beam), $\lambda_r = 1$ Å (hard x-ray!)

Longitudinal Lorentz factor $\gamma_z \equiv \frac{1}{\sqrt{1-\beta_z^2}}$

where $\beta_z \equiv v_z / c$

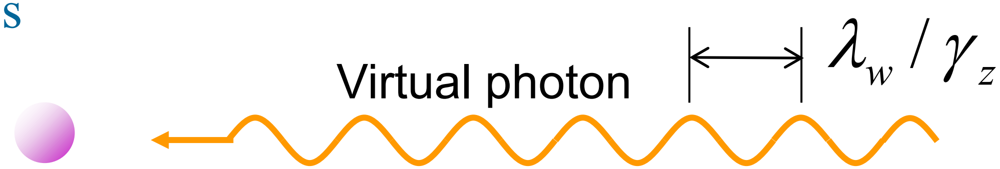
# Undulator Radiation

## In laboratory frame



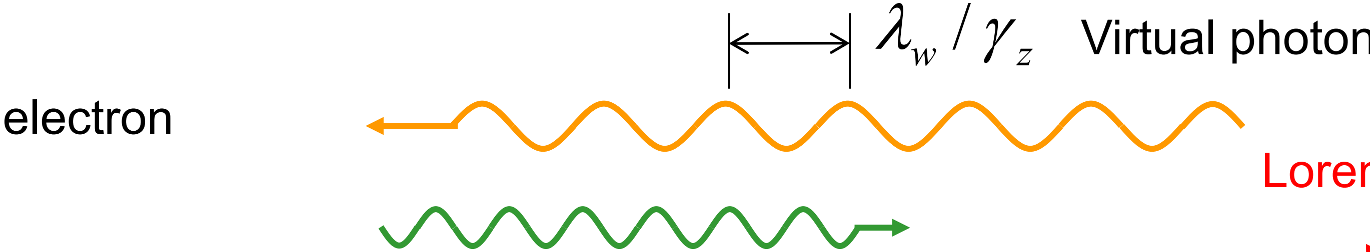

In electron rest frame, the electron sees a "wave" with fields:

$$\vec{E}' = \gamma\vec{\beta}\times\vec{B}, \vec{B}' = \gamma\vec{B}$$



# Spontaneous Undulator Radiation

## Electron Rest Frame




Lorentz contraction

Electron oscillation frequency in the electron frame

$$f' = (T')^{-1} = \left( \frac{\lambda_w \boxed{/\gamma_z}}{v_z} \right)^{-1}$$

## Laboratory Frame

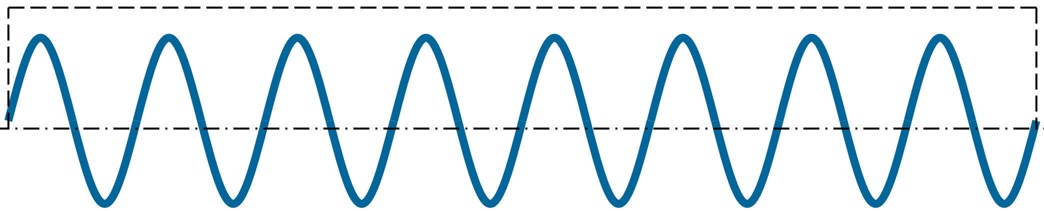

Doppler Shift $\quad f = f' \sqrt{\dfrac{1+\beta_z}{1-\beta_z}}$ $\quad \Rightarrow \quad$ $\boxed{\lambda = \lambda_w \left( \dfrac{1}{\beta_z} - 1 \right) \approx \dfrac{\lambda_w}{2\gamma_z^2}}$

For $\lambda_w \sim 1$ cm, 100 MeV( $\gamma_z$ ~ 200), $\Rightarrow \lambda$ = 125 nm

"Cheap" long-wavelength virtual photon $\Rightarrow$ expensive short-wavelength photon

# Spontaneous Undulator Radiation

$$\tau = \frac{L_r}{c}, L_r = N_w \lambda_r$$



$N_w$: *number of undulator periods*
$L_r$: slippage distance
$\tau$: radiation pulse length
$\omega_r$: resonant radiation frequency

$$e^{i\omega_r t} \times \mathrm{rect}[\frac{t}{\tau}]$$

## Radiation Spectrum

### Lemma

Rectangular function: $\mathrm{rect}[t] = \begin{cases} 1 & \text{for } |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$

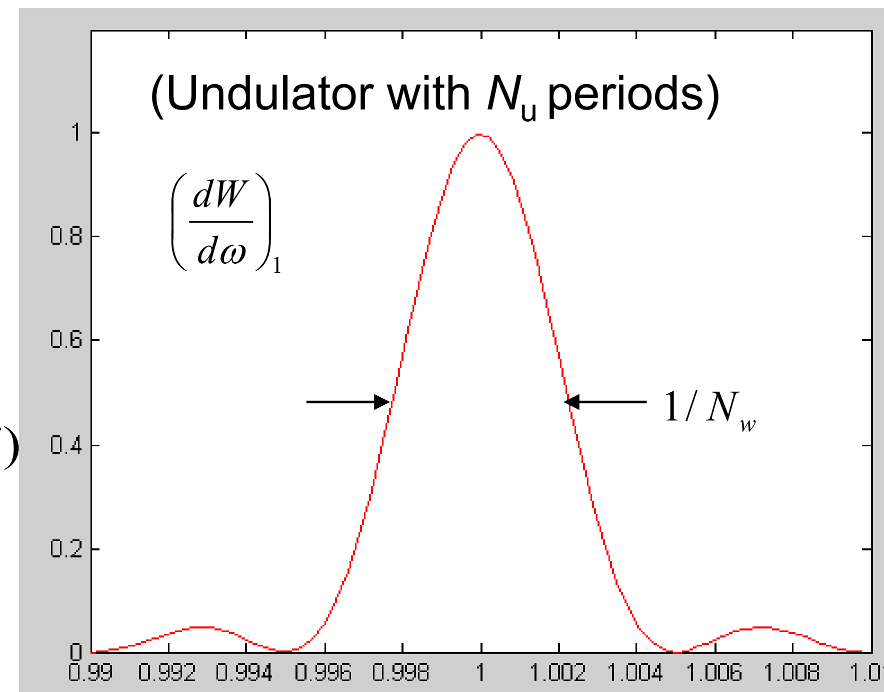
Fourier Transform $\{\mathrm{rect}[t]\} = \dfrac{\sin(\omega/2)}{\omega/2} = \mathrm{sinc}(f)$

### So,

Fourier Transform $\{e^{i\omega_r t} \times \mathrm{rect}[\dfrac{t}{\tau}]\}$

$$\propto \frac{\sin[2N_w(\omega/\omega_r - 1)]}{2N_w(\omega/\omega_r - 1)}$$

### Spectral-energy Lineshape Function




$$\omega/\omega_r$$

Spectral energy $\left(\dfrac{dW}{d\omega}\right)_1 \propto \{\mathrm{sinc}[2N_w(\omega/\omega_r - 1)/\pi]\}^2$

# Effect of Magnetic field on e⁻ Quiver Motion

A general assumption: a relativistic beam $\gamma >> 1$

Assume a planar/linear wiggler with a wiggler field of $\vec{B} = \hat{y}\sqrt{2}B_{rms}\sin k_w z$

Begin with the Lorentz force equation $\dfrac{d\vec{p}}{dt} = e\vec{v}\times\vec{B}$ , where $\vec{p} = \gamma m_0 \vec{v}$

$$v_x \approx \frac{-\sqrt{2}ca_w}{\gamma}\cos(k_w z) = \frac{-\sqrt{2}ca_w}{\gamma}\cos(\underline{k_w v_z t})$$

Wiggler wavenumber

$$\boxed{k_w = 2\pi/\lambda_w}$$

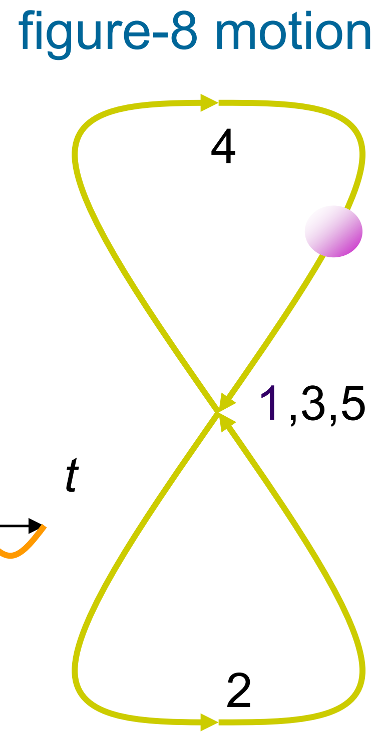
$$v_z = \sqrt{v^2 - v_x^2} \approx v - \frac{a_w^2}{2\gamma^2}\frac{c}{\beta}\cos(2k_w z)$$

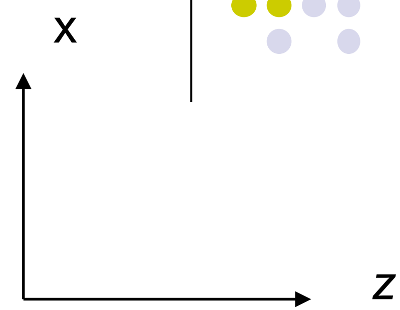
$$= v - \frac{a_w^2}{2\gamma^2}\frac{c}{\beta}\cos(\underline{2k_w v_z t})$$

√

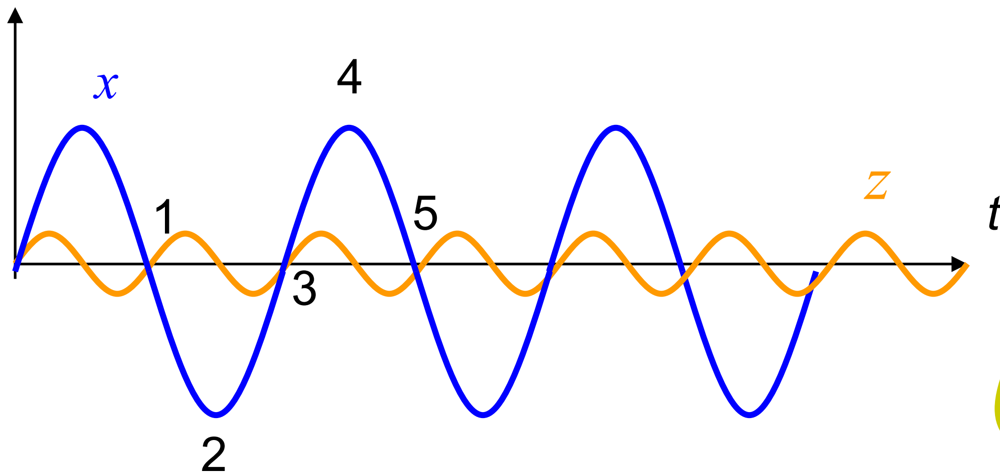
where $a_w = \dfrac{eB_{rms}}{m_0 c k_w}$

Wiggler parameter



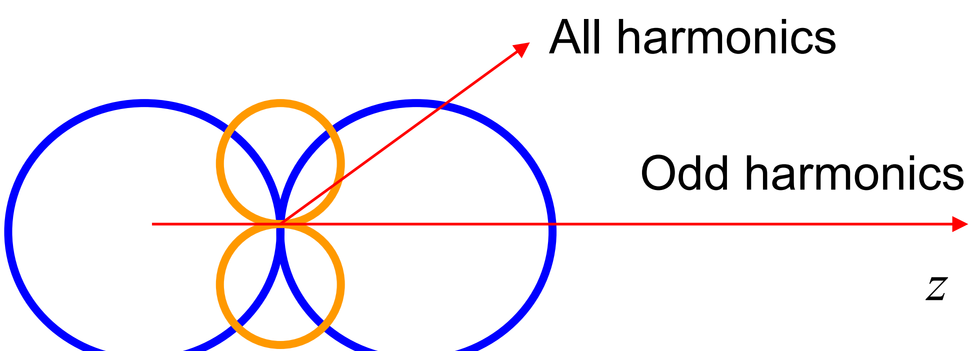




# In the Electron Rest Frame



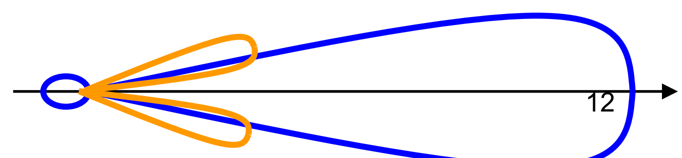

$\vec{v}_{z,2\omega} \times \vec{B}_{y,\omega}$ generates $\vec{v}_{x,3\omega}$, $\vec{v}_{x,3\omega} \times \vec{B}_{y,\omega}$ generates $\vec{v}_{z,4\omega}$ ……..

**Dipole radiation pattern in the electron rest frame**




**Dipole radiation pattern in the laboratory frame**

($\gamma$ >> 1)



# Undulator Radiation Wavelength

Because

$$\gamma_z \equiv \frac{1}{\sqrt{1-\beta_z^2}} = \frac{1}{\sqrt{1-v_z^2/c^2}}, \text{ and } \quad \lambda \approx \frac{\lambda_w}{2\gamma_z^2}$$

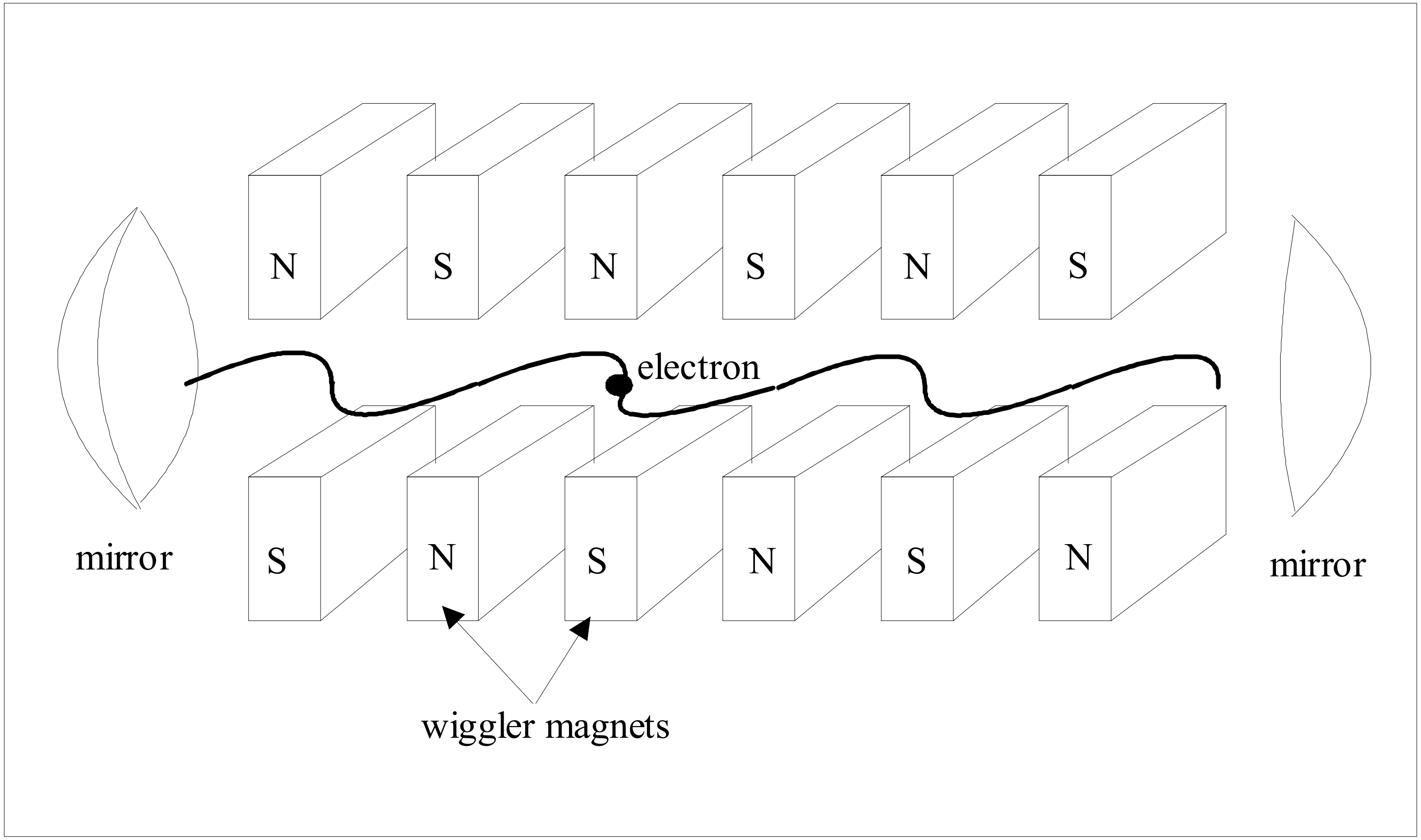
$$\Rightarrow \frac{1}{\gamma_z^2} = \frac{1+a_w^2}{\gamma^2} \quad \text{where} \quad a_w = 0.093 B_{rms}(\text{kgauss}) \times \lambda_w(\text{cm})$$

is called the *wiggler/ undulaotor parameter*

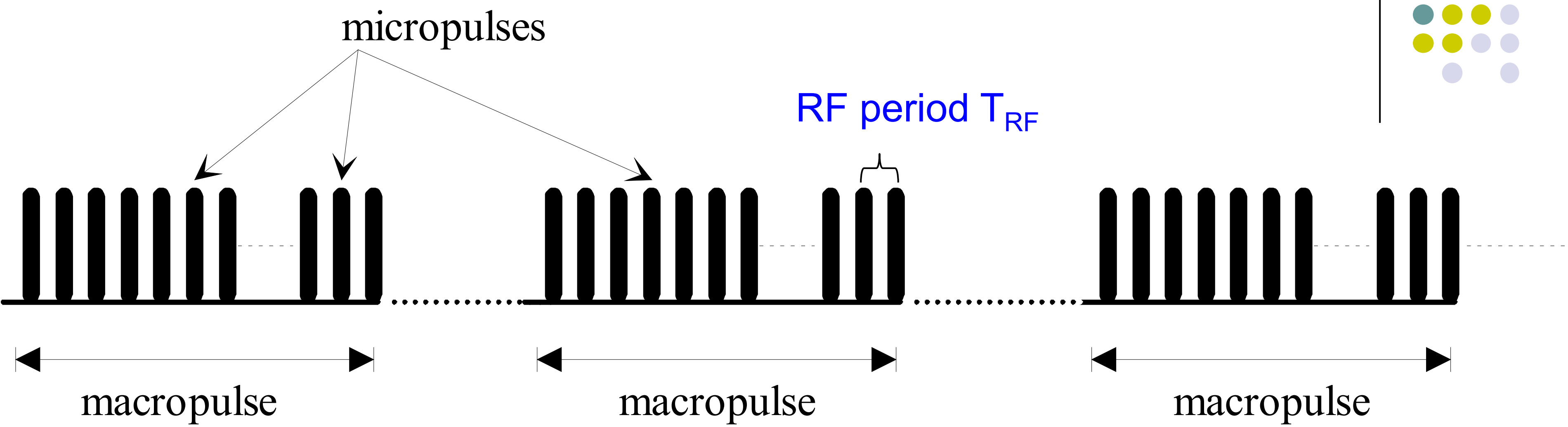
$$\Rightarrow \boxed{\lambda = \frac{1+a_w^2}{2\gamma^2}\lambda_w \quad \text{(FEL synchronism condition)}}$$

Undulator radiation wavelength can be tuned by magnetic field *B*, wiggler period $\lambda_w$, and electron energy $\gamma$

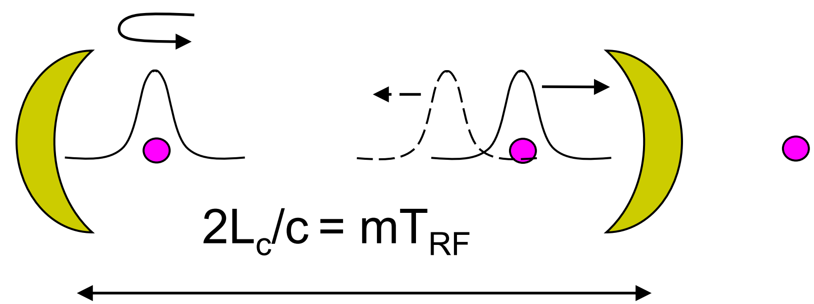
# A Free-electron Laser Oscillator



# Pulse Structure of a RF Linac-driven FEL Oscillator




Taking SLAC S-band RF as an example:
1. RF frequency = 2.856 GHz
2. Micropulse length ~ 10 ps
3. Macropulse length: 1-5 μs
4. Macropulse repetition rate: 10-100 Hz




* Macropulse length > laser buildup time

**Oscillation condition:** $E_0 e^{-j\phi} e^{g_{th}L_c - 2\alpha L_c} = E_0$

$g_{th}L_c$: 1-way threshold gain, $2\alpha L_c$: roundtrip loss, $\phi$ = roundtrip phase

(1) Threshold condition: gain = loss $g_{th} = 2\alpha$ (2) Phase condition: $\phi = 2m\pi$

# Electron-Wave Energy Exchange

$$\frac{dK}{dt} = e\vec{v}\cdot\vec{E}$$ $K$: electron kinetic energy

Wave Amplification $$\Delta W = \int \vec{F}\cdot\vec{v}dt = e\int_{\tau=L/v_{//}} \vec{E}\cdot\vec{v}dt < 0$$

Particle Acceleration $$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}\cdot\vec{v}dt > 0$$

Transverse Coupling
(fast wave, Eg. Compton/Thomson/undulator radiation etc.)

$$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{\perp}\cdot\vec{v}_{\perp}dt$$

Longitudinal Coupling
(slow wave, Eg. Smith-Purcell radiator, Traveling wave tube, backward-wave oscillator etc.)

$$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{//}\cdot\vec{v}_{//}dt$$

# Resonant Interaction between Electron and Field

To have FEL gain

$$\boxed{\Delta W = e\int_{\tau=L_w/v_z} \vec{E}\cdot\vec{v}\,dt < 0}$$

$L_W$ is the length of the wiggler

For $E_x = E_0\cos(\omega t - kv_z t + \phi)$ and $v_x = \dfrac{-\sqrt{2}c_0 a_w}{\gamma}\cos(k_w v_z t)$

$$\Rightarrow \vec{E}\cdot\vec{v} \propto \cos\{[k-(k+k_w)\beta_z]ct+\phi\} + \cos\{[k-(k-k_w)\beta_z]ct+\phi\}$$

**$\Psi$:pondermotive phase** **fast wave**

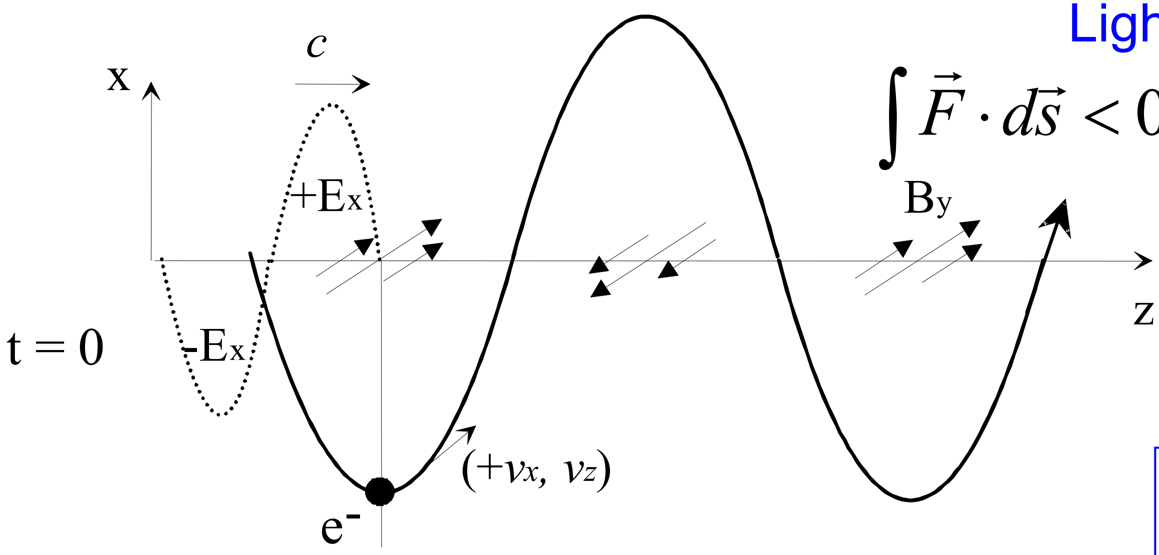
whether $\vec{E}\cdot\vec{v} > 0$ (radiation) or $\vec{E}\cdot\vec{v} < 0$ (particle acceleration) depends on $\phi$

To have appreciable value in $\int_{\tau=L_w/v_z} \vec{E}\cdot\vec{v}\,dt$

$$k-(k+k_w)\beta_z = 0 \Rightarrow \lambda = \frac{1+a_w^2}{2\gamma^2}\lambda_w$$ The FEL synchronism condition

$$k-(k-k_w)\beta_z = 0 \Rightarrow \beta_z \equiv v_z/c_0 > 1$$ Impossible in vacuum

# Undulator Light Amplification




Light amplification

$$\int \vec{F}\cdot d\vec{s} < 0 \text{ or } \int \vec{v}\cdot\vec{E}dt > 0$$

$$\Delta W = e\int_{\tau=L/v_{//}} \vec{E}_{\perp}\cdot\vec{v}_{\perp}dt$$

$$\tau = \boxed{\frac{\lambda_w/2+\lambda_z/2}{c} = \frac{\lambda_w/2}{v_z}}$$

light slips one wavelength ahead per wiggler period

$$\boxed{\lambda = \lambda_w\left(\frac{1}{\beta_z}-1\right) \approx \frac{\lambda_w}{2\gamma_z^2}}$$

# Pendulum Equation

The pondermotive (beat) phase $\psi = (k + k_w)z - \omega t$

was previously found from the beam-wave energy coupling equation

$$\frac{dK}{dt} = ev_x E_x = \frac{ec_0 a_w E_0}{\sqrt{2}\gamma}\cos\{\omega t - (k + k_w)z(t) + \phi\}$$

Take first derivative of $\psi$ with respect to z and use the FEL synchronism condition to obtain

$$\frac{d\psi}{dz} = 2k_w \frac{\gamma - \gamma_r}{\gamma_r} = 2k_w \boxed{\frac{\Delta\gamma}{\gamma_r}}$$

where $\gamma_r$ is the resonant particle energy satisfying the synchronism condition

$$\lambda = \lambda_w \frac{1 + a_w^2}{2\gamma_r^2} \qquad \text{or} \qquad k_w = k\frac{1 + a_w^2}{2\gamma_r^2}$$

A second derivative to the beat phase with respect to *z* gives the

pendulum equation

$$\boxed{\frac{d^2\psi}{dz^2} = -k_\psi^2 \sin\psi}$$

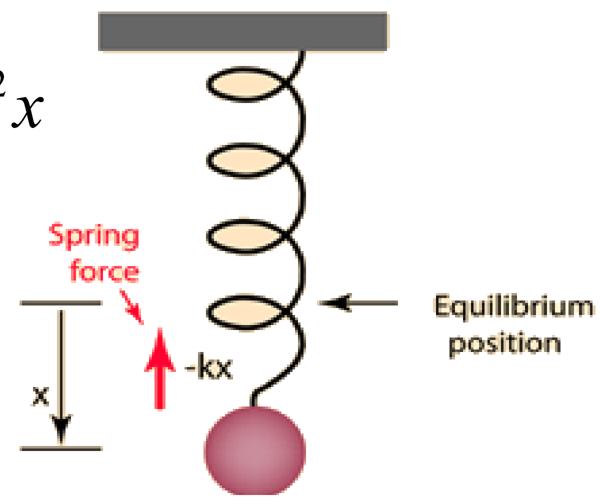
where $$k_\psi^2 = \left[\frac{e}{\gamma_r m_0 c_0}\right]^2 \frac{\sqrt{2} B_{rms} E_0}{c_0} \equiv \frac{2\pi}{L_\psi}$$

$L_\psi$: synchrotron oscillation wavelength

For a small $\Psi$, $$\frac{d^2\psi}{dz^2} \sim -k_\psi^2 \psi$$

Recall the harmonic oscillator equation

$$\frac{d^2 x}{dt^2} = -\omega_0^2 x$$

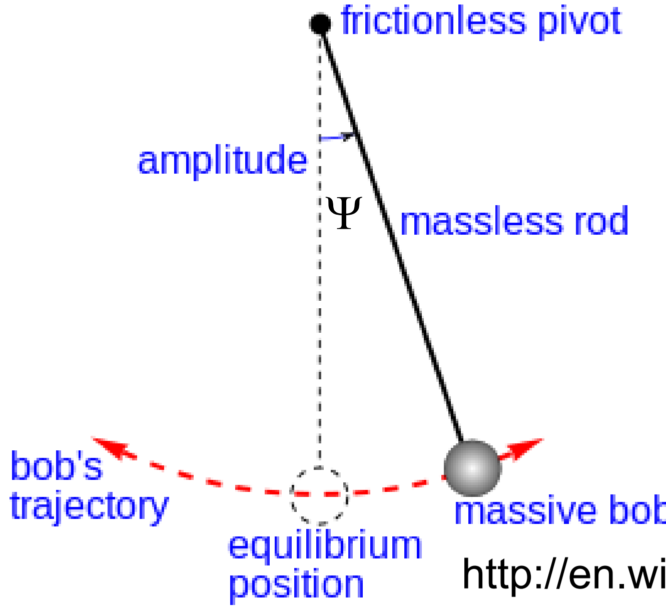



Particles oscillate, drift in the pondermotive phase .

http://hyperphysics.phy-astr.gsu.edu/hbase/oscda.html

# pendulum equation

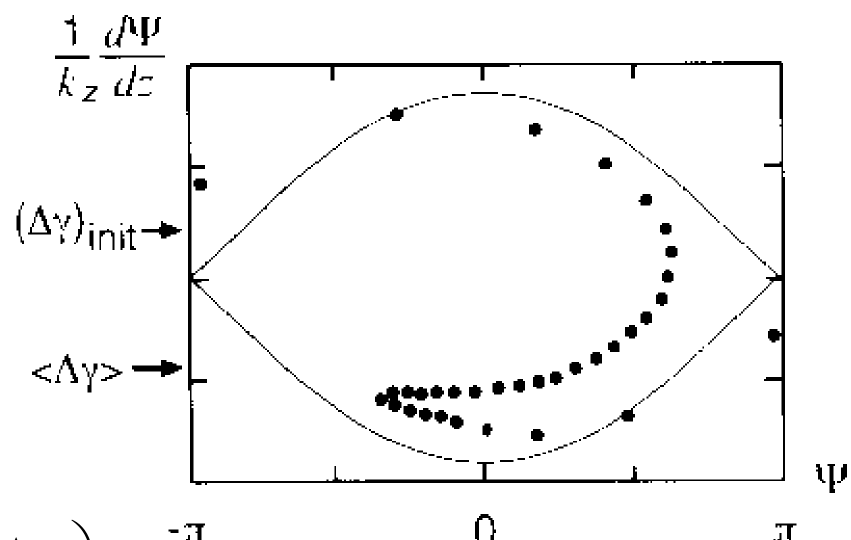
$$\boxed{\frac{d^2\psi}{dz^2} = -k_\psi^2 \sin\psi}$$




http://en.wikipedia.org/wiki/Pendulum

With the definition of $k_\psi$, the *phase diagram* can be plot from

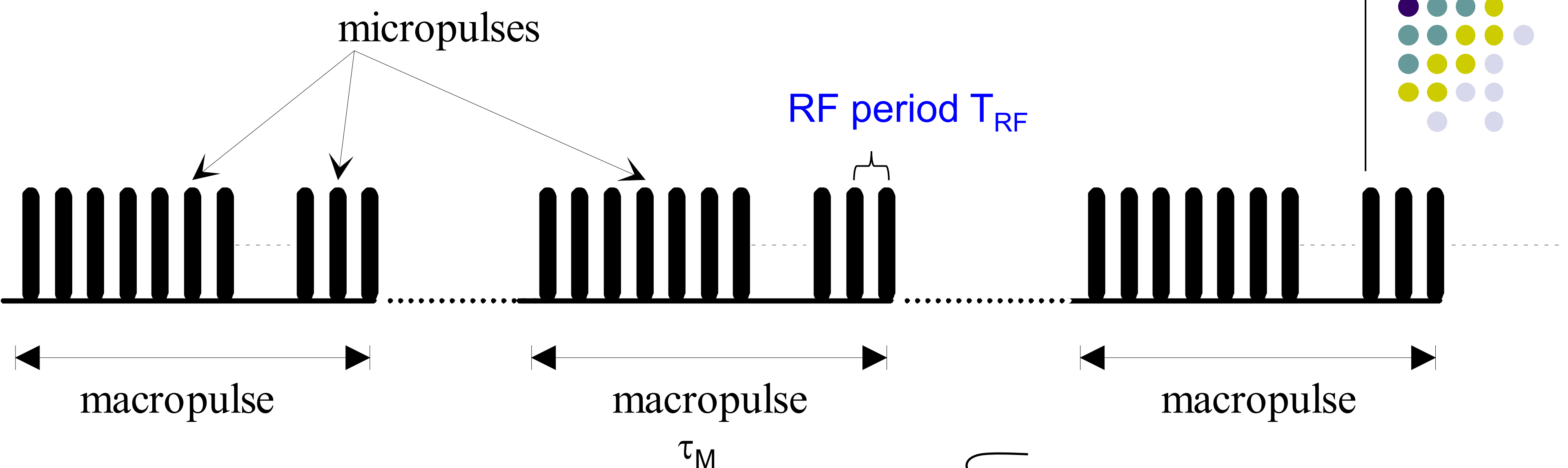
$$\frac{d\psi}{dz} = \pm\sqrt{2}k_\psi\sqrt{\cos\psi + 1} = 2k_w\frac{\Delta\gamma}{\gamma_r}$$

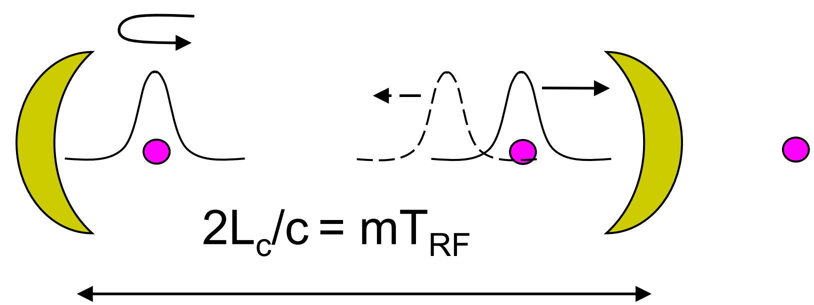



The bucket height = 4 $k_\psi$, and the maximum energy extraction occurs at half synchrotron wavelength: FEL length is ~$L_\psi$/2

The maximum energy efficiency for an FEL = $\left(\frac{\Delta\gamma}{\gamma_r}\right)_{\max} = 1/(2N_w)$

# FEL Buildup Time $\tau_B$




$$\overbrace{P_b \times \left[\frac{1}{2N_w}\right]}^{\text{Saturation power}} = P_s \times \left[\overbrace{e^{(gL_c - 2\alpha L_c)}}^{\text{Roundtrip net gain}}\right]^{\frac{\tau_B}{2L_c/c}}$$

Beam power

Spontaneous radiation power

# of roundtrips

$$\left(\frac{\Delta\gamma}{\gamma_r}\right)_{\max} = 1/(2N_w)$$

* Macropulse length > laser buildup time

$$\boxed{\tau_M > \tau_B}$$

# FEL Gain

$$G = \frac{W_f - W_i}{W_i} = \frac{\Delta W}{W_i} = e^{gL_c}$$

To have gain

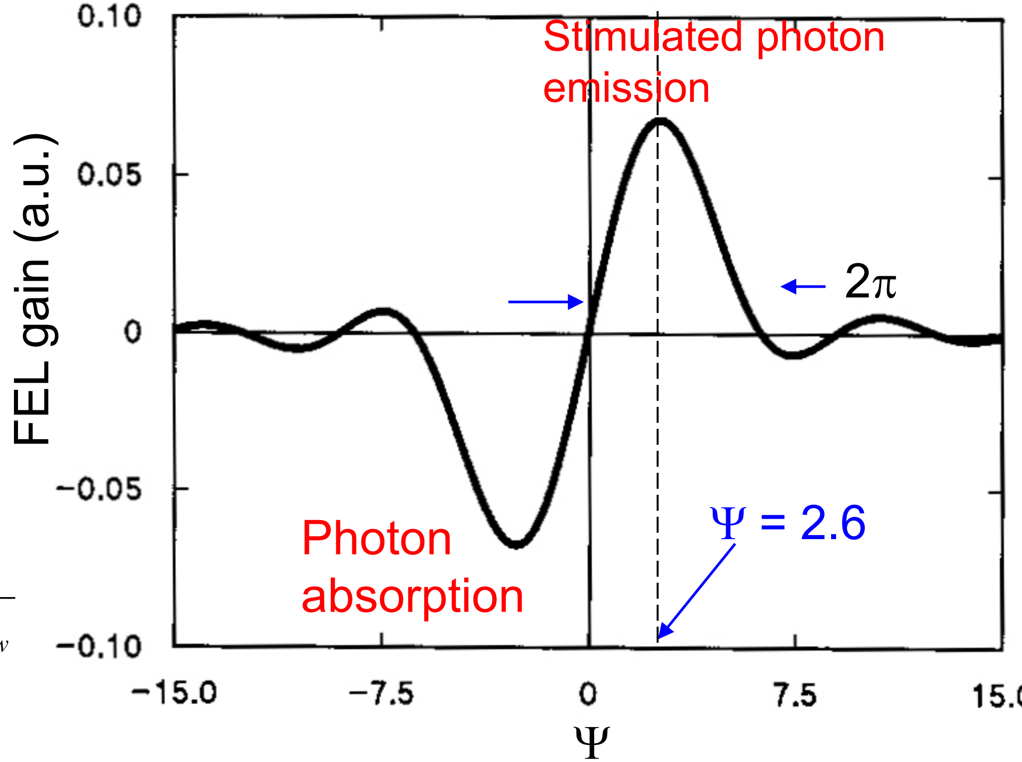
$$\boxed{\Delta W = e\int_{\tau = L_w / v_z} \vec{E}\cdot\vec{v}\,dt < 0}$$

$L_W$ is the length of the wiggler

For $E_x = E_0 \cos(\omega t - k v_z t + \phi)$ and $v_x = \frac{-\sqrt{2}ca_w}{\gamma}\cos(k_w v_z t)$

whether $\vec{E}\cdot\vec{v} > 0$ (radiation) or $\vec{E}\cdot\vec{v} < 0$ (particle acceleration) depends on $\phi$

- **Gain is small for short wavelength FEL**
- **Electron injection energy has to be detuned from synchronism**
- **Gain is proportional to injection current**
- **Energy spread can't exceed** $\frac{\Delta\gamma}{\gamma} < \frac{1}{2N_w}$ **where $N_w$ is the number of wiggler period**



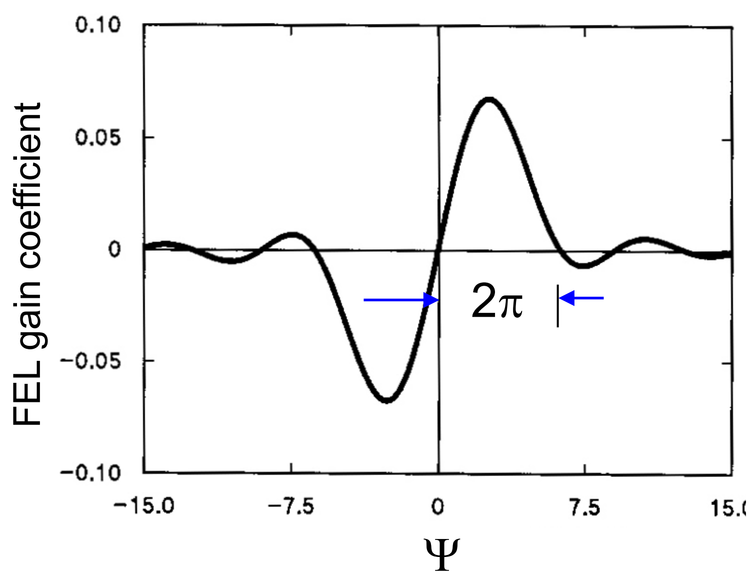
# Energy Spread Requirement

Refer to the FEL gain curve, for an electron to contribute its energy to the FEL gain, the acceptance phase width has to be confined to $2\pi$ or

$$\Delta\Psi = \left| [\omega - (k+k_w)\bar{v}_z] \frac{L_w}{\bar{v}_z} \right|_{\max} = 2\pi \Rightarrow \left| [\frac{\omega}{\bar{v}_z} - (k+k_w)] L_w \right|_{\max} = 2\pi$$

$$\Rightarrow \left| \frac{d\psi}{dz} L_w = 2k_w \frac{\Delta\gamma}{\gamma_r} L_w \right| < 2\pi \Rightarrow \boxed{\left| \frac{\Delta\gamma}{\gamma_r} \right| < \frac{1}{2N_w}}$$

where $N_w$ is the number of undulator periods




So, the energy spread of the electron beam for an FEL has to be less than $1/(2N_w)$

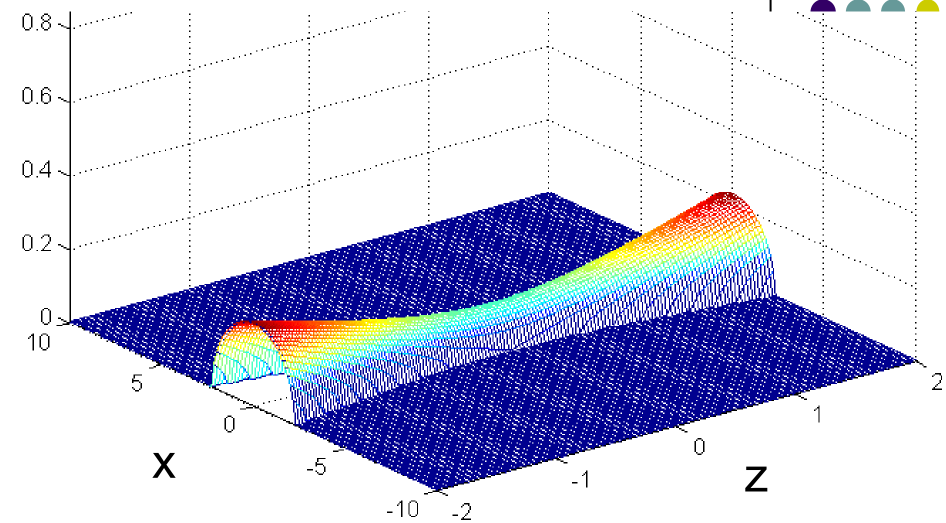
# Emittance Requirement for an FEL

## A Gaussian Laser Beam

Rayleigh range $z_{o,R} = \dfrac{\pi w_0^2}{\lambda}$

Far-field diffraction angle = $\theta \sim \dfrac{w_0}{z_R}$

The phase space (angle and beam size) area is $\pi \theta w_0 \sim \lambda$



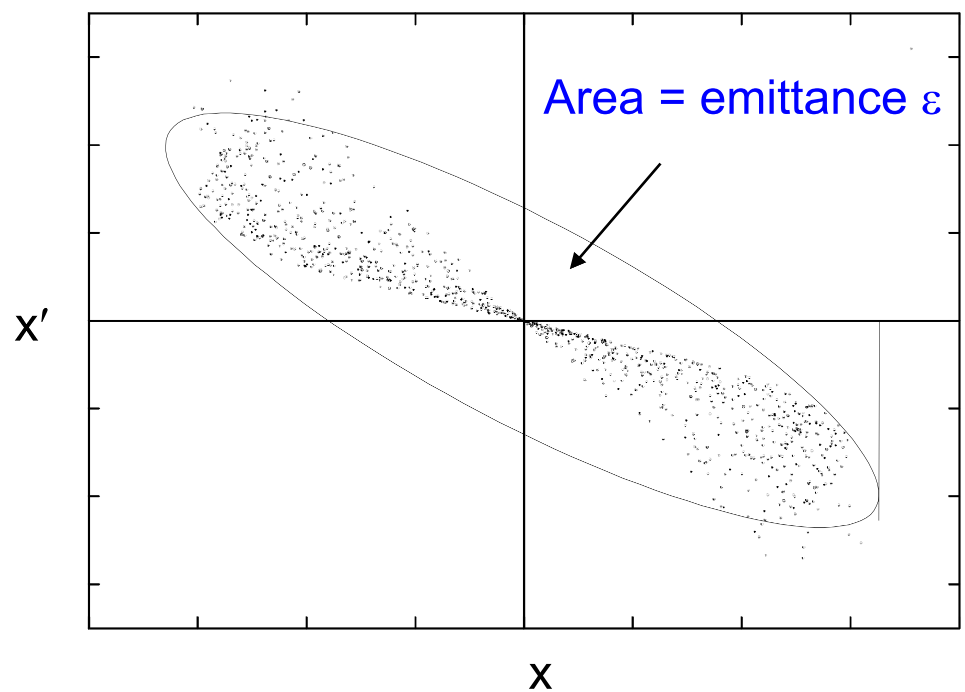

## An Electron Beam

The phase space area is the beam's geometric emittance $\varepsilon$

To place an electron beam
Inside an optical beam

$$\varepsilon < \lambda$$

Therefore long-wavelength FEL is more forgiving to e-beam quality



# FEL Gain Bandwidth

The spectral bandwidth is defined by the variation of the spectral ratio

$$\left|\frac{\Delta\lambda}{\lambda}\right| = \left|\frac{\Delta\omega}{\omega}\right|$$

within the half width of the gain curve

$$\Delta\Psi = \left|\Omega\tau = [\omega - (k + k_w)\bar{v}_z]\frac{L}{\bar{v}_z}\right| < \pi$$

From the FEL synchronism condition $\lambda = \lambda_w \frac{1+a_w^2}{2\gamma^2}$, it is straightforward to show

$$\left|\frac{\Delta\lambda}{\lambda}\right| = 2\left|\frac{\Delta\gamma}{\gamma}\right|$$

However the maximum allowed $\Delta\gamma/\gamma < 1/(2N_w)$ is obtained from the full width. For a half width

$$\left|\frac{\Delta\lambda}{\lambda}\right| = 2\left|\frac{\Delta\gamma}{\gamma}\right| < 2 \times \frac{1}{2N_w} \times \frac{1}{2} = \boxed{\frac{1}{2N_w}}$$

# Characteristics of a Free-electron Laser

1. Laser: a coherent light source

2. Wavelength tunable:
   by varying the magnetic field and the electron energy

3. High peak power: GW-MW in 0.1~ 10 psec micropulse

4. High average power: kW in > ~ μsec macropulse

# General Requirements for Building an FEL

Gain > loss

In particular
i. Electron energy spread $\Delta\gamma/\gamma < 1/2N_w$
ii. Electron emittance $\varepsilon < \lambda$