Introduction to High Gain Free Electron Lasers

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Outline

- Introduction to synchrotron radiation and FELs
- 1D theory of high gain FELs
- Self-amplification of spontaneous radiation (SASE)
- Seeded FELs
  - Self-seeding
  - High-gain harmonic-generation (HGHG) and other seeding schemes
- 3D and beam quality effects
- High gain FEL facilities and applications

Basic Postulates of Special Relativity

- *Postulate of Relativity* – the law of nature and the results of all experiments performed in a given frame of reference are independent of the translational motion of the whole system.
- *Postulate of the constancy of the speed of light* – the speed of light is finite and independent of the motion of its source. In other words, in every inertial frame, there is a finite universal limiting speed c for physical entities (although it is a postulate, it is based on our experimental observations).

Lorentz Transformation

Lorentz transformation of '*space-time*' coordinates can be deduced directly from the two basic postulates of special relativity:

$$\begin{cases} x' = \dfrac{x - vt}{\sqrt{1 - \left(\dfrac{v^2}{c^2}\right)}} \\ y' = y \\ z' = z \\ t' = \dfrac{t - \left(\dfrac{v}{c^2}\right)x}{\sqrt{1 - \left(\dfrac{v^2}{c^2}\right)}} \end{cases}$$

Length contraction: a body's length is measured to be longest when it is at its rest frame.

$$\Delta x = \Delta x' \sqrt{1 - \left(\frac{v^2}{c^2}\right)}$$

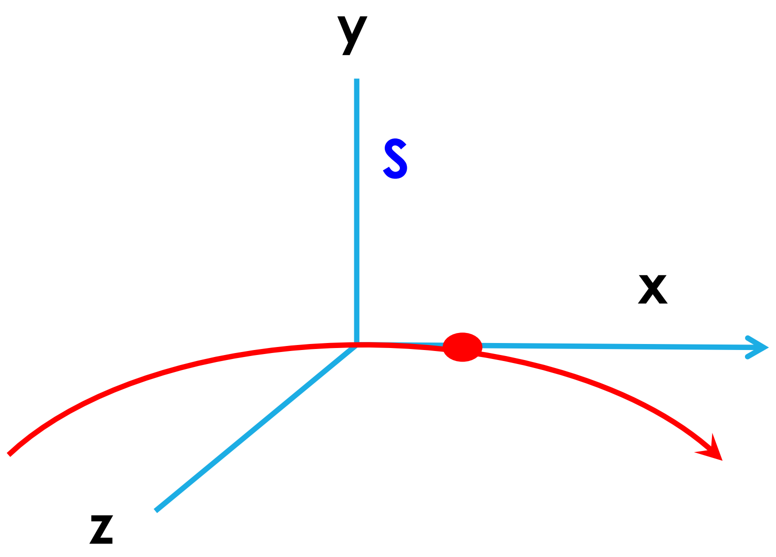
Time dilatation: a clock is measured to go at fastest rate when it is at its rest frame.

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}$$

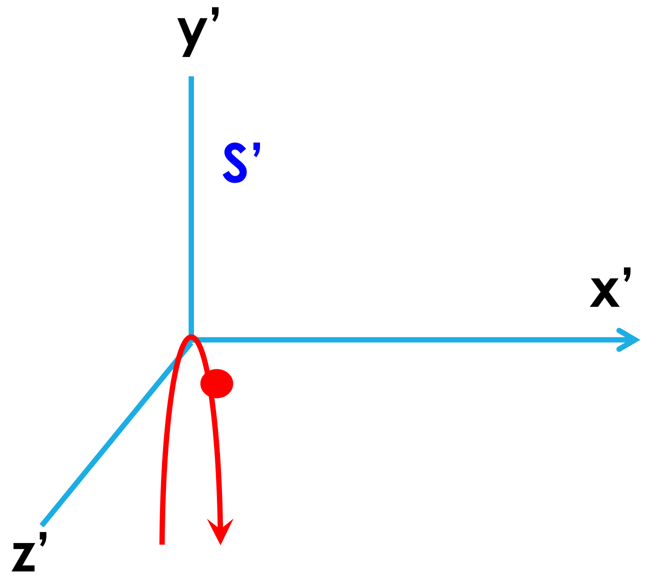
" A thousand years in your sight
are like a day that has just gone by,
or like a watch in the night."

Psalm 90:4

Orbit in Electron's Reference Frame



laboratory frame

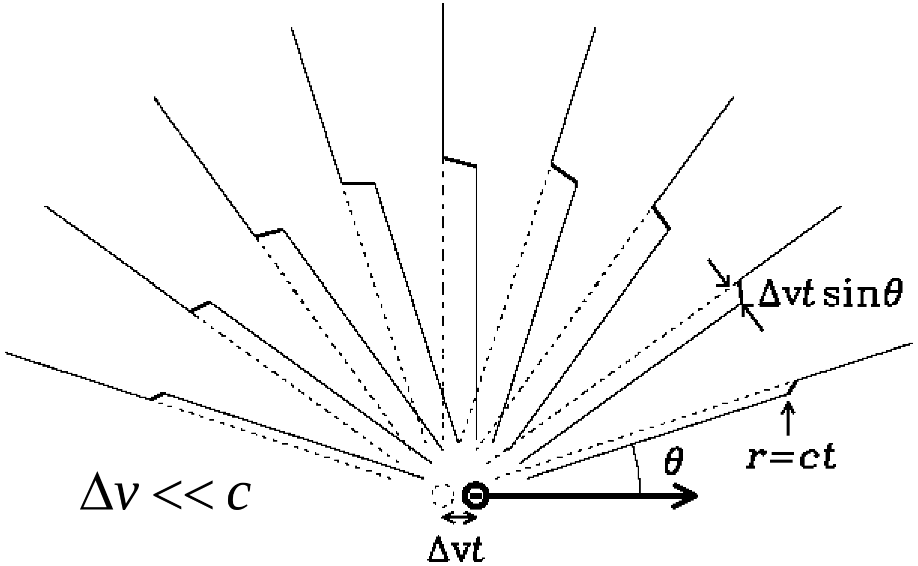


electron orbit seen in reference frame moving at particle tangential velocity

$$v'_x, v'_y << v'_z \approx c$$

Radiation by a Slow Particles

Angular distribution of the radiation power by a slowly moving (i.e. v << c) charged particle under acceleration:



$$\Delta v << c$$

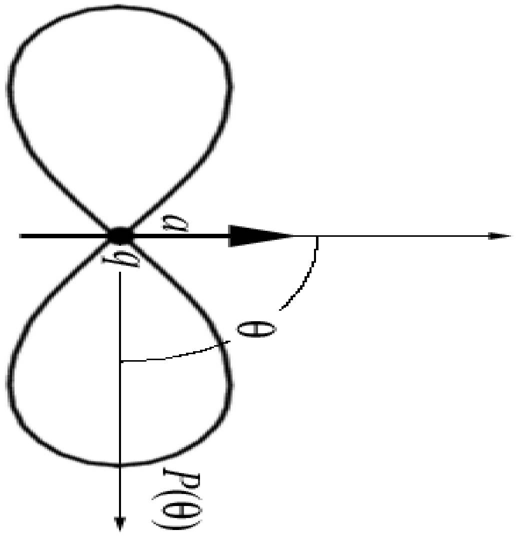
$$\frac{E_\perp}{E_r} = \frac{\Delta vt \sin\theta}{ct} \quad \rightarrow \quad E_\perp = \frac{1}{4\pi\varepsilon_0}\frac{q\dot{v}\sin\theta}{rc^2}$$

instant energy flow

$$\left|\vec{S}\right| = \left|\vec{E}_\perp \times \vec{H}_\perp\right| = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left|\vec{E}_\perp\right|^2 = \frac{1}{16\pi^2\varepsilon_0}\frac{q^2\dot{v}^2\sin^2\theta}{r^2c^3}$$

$$\frac{dP'}{d\Omega} = \left|\vec{S}\right| = \frac{1}{16\pi^2\varepsilon_0}\frac{e^2a'^2}{c^3}\sin^2\theta$$

power radiated into a unit solid angle by an electron



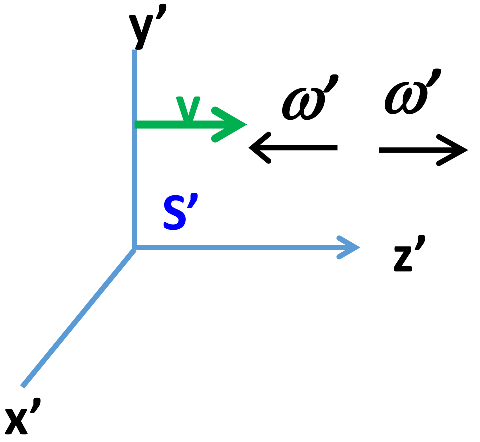
$$P' = \frac{e^2a'^2}{6\pi\varepsilon_0c^3}$$

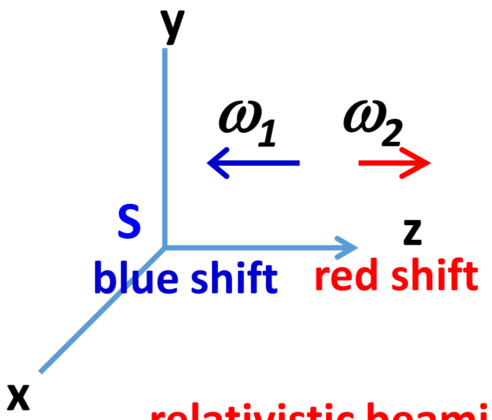
Larmor formula

Relativistic Effects

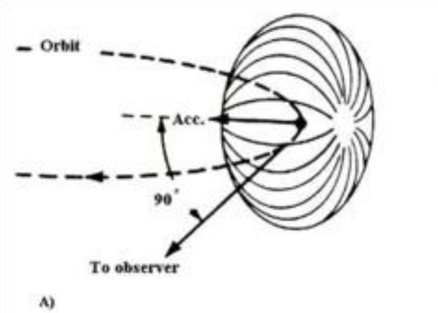
Lorentz transformation

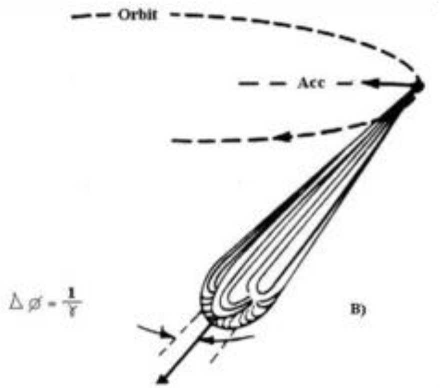
$$\begin{cases} \tan\theta = \dfrac{\sin\theta'}{\gamma(\cos\theta' + \beta)} \\ \omega = \gamma\omega'(1+\beta\cos\theta') \end{cases} \longrightarrow \theta \approx \frac{1}{\gamma} \quad \textit{for large } \gamma$$





relativistic beaming





$$\hbar\omega_1 + \hbar\omega_2 = 2\gamma\hbar\omega'$$

relativistic Doppler effect

$$P\tau = \gamma P'\tau' = \gamma P'\tau/\gamma$$

time dilatation

$$\boxed{P = P'}$$

- total radiated power in laboratory frame is the same as that in the frame moving tangential to the particle trajectory
- the power loss due to Doppler shift in one direction is gained back in the other direction.

Power of Synchrotron Radiation by an Electron

$$P = P' = \frac{e^2 a'^2}{6\pi\varepsilon_0 c^3}$$

but $a' = \gamma^2 a$ and
centripetal acceleration $a = c^2/\rho$

$$P = \frac{1}{6\pi\varepsilon_0}\frac{e^2 c}{\rho^2}\gamma^4$$

total radiation (*synchrotron radiation*) loss per turn in a synchrotron

$$U_0 = \int_0^{2\pi R} P ds / c = C_\gamma E^4 R \left\langle \frac{1}{\rho^2} \right\rangle$$

$$\langle P \rangle = U_0 / T_0$$

$$T_0 = 2\pi R / c$$

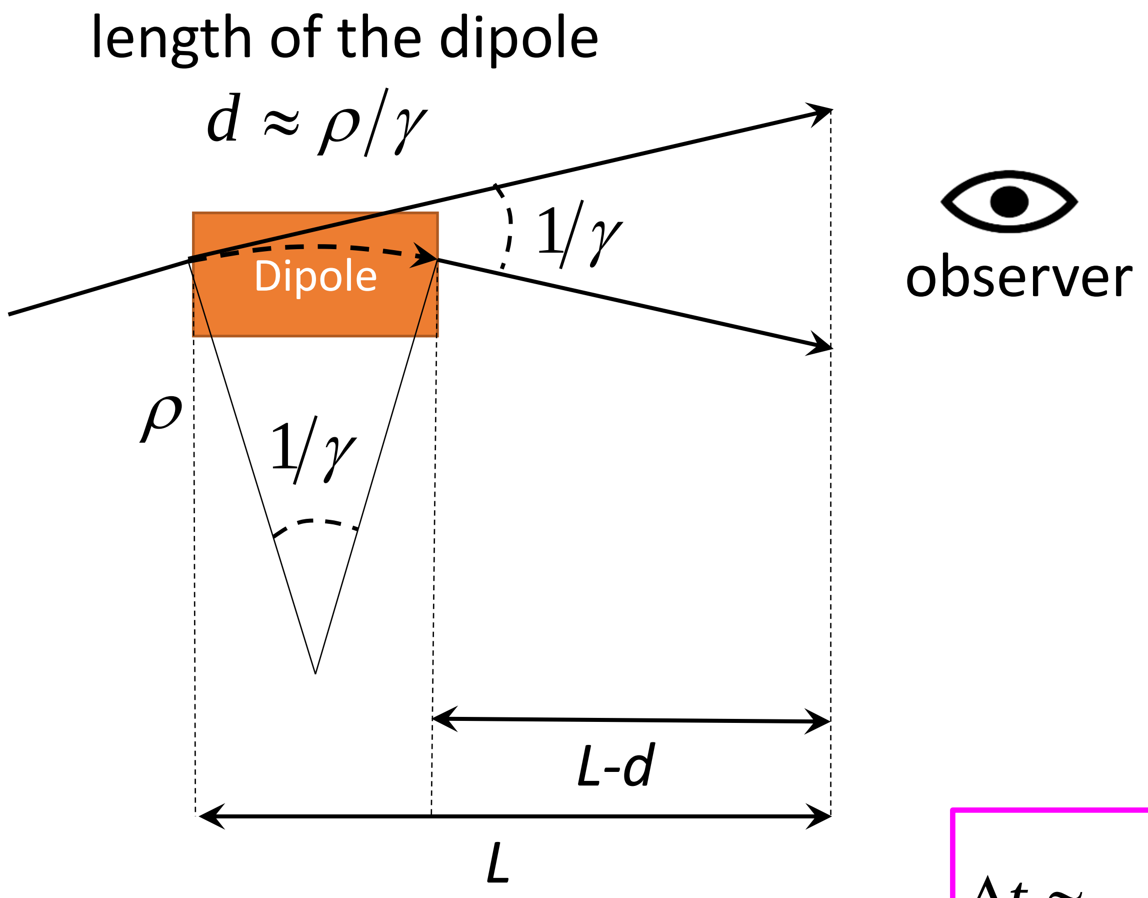
where

$$C_\gamma = \frac{4\pi}{3}\frac{r_0}{(mc^2)^3} = 8.846\times 10^{-5}\frac{meters}{GeV^3}$$

$$r_0 = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{mc^2} = 2.818\times 10^{-15} m$$

is the classical radius

Pulse Duration of SR from Bending Magnet



The pulse duration of radiation emitted by an electron as detected by the observer is:

$$\Delta t = t_1 - t_0$$

$$= \left(\frac{d}{v} + \frac{L-d}{c}\right) - \frac{L}{c}$$

$$= \frac{d}{v} - \frac{d}{c} = \frac{d}{c}\left(\frac{1}{\beta} - 1\right) \approx \frac{d}{c}\frac{1}{2\gamma^2}$$

$$\Delta t \approx \frac{\rho}{2c\gamma^3}$$

For a 3 GeV storage ring with bending magnets of 10-m in radius of curvature. Pulse duration of SR emitted by an electron is about 8 x10^{-20}s !!

Typical Spectrum of SR from Bending Magnet

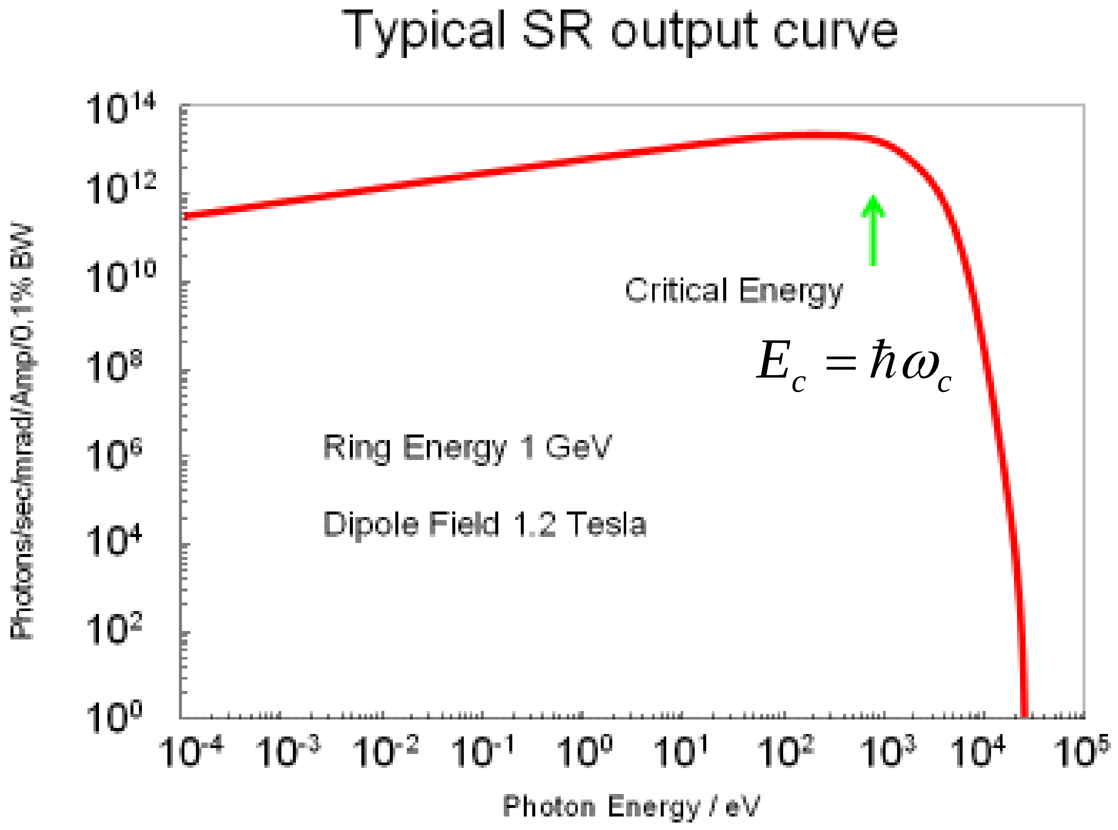
Due to the short radiation pulse duration, the spectrum of SR from a bending magnet is broad. The maximum frequency of the spectrum can be estimated as:

$$\omega_c \approx 1/\Delta t$$

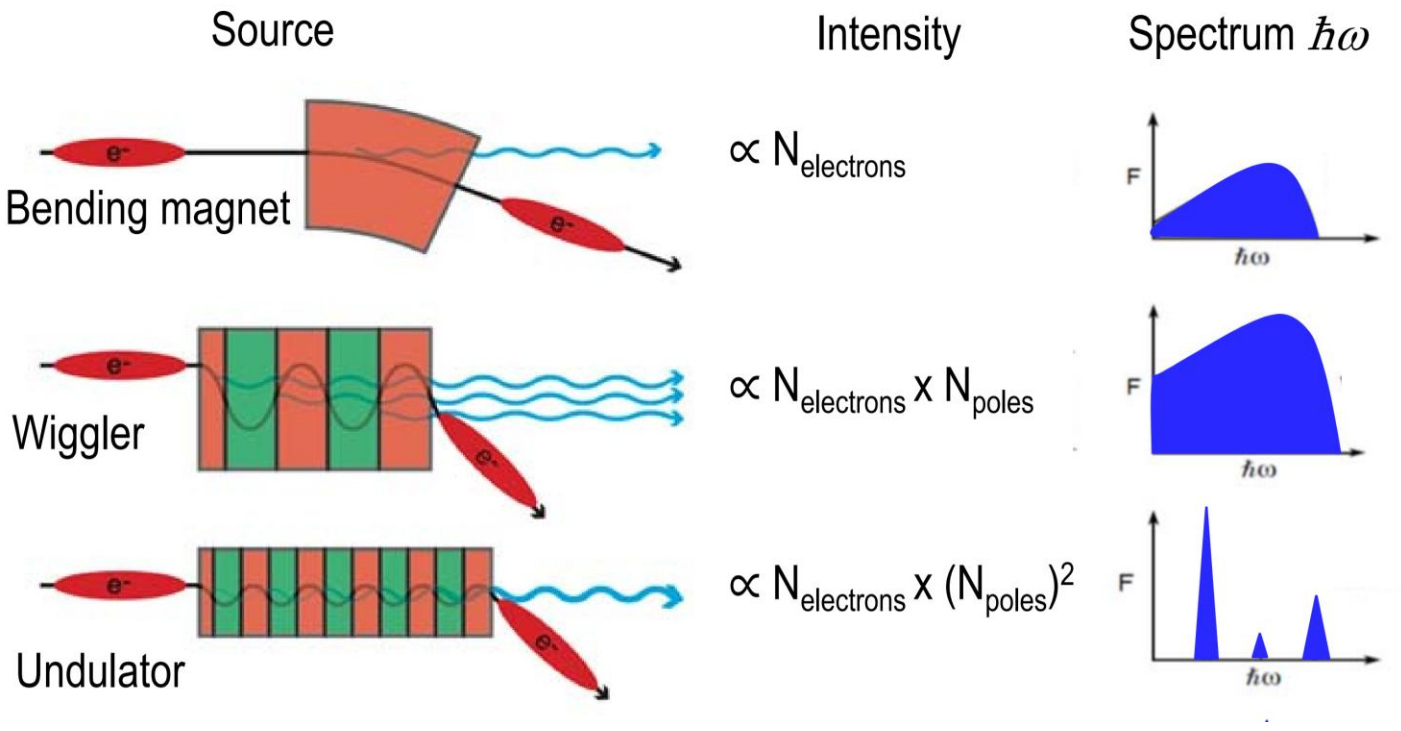
$$\omega_c \approx 2c\gamma^3/\rho \approx 2\omega_0\gamma^3$$

ω_0 is the revolution frequency of electron going around the ring.

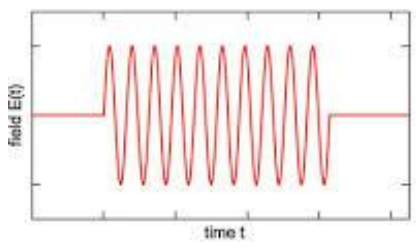
For TPS, a 3-GeV ring, the revolution is 578.3 kHz. The maximum frequency (critical frequency) of SR with appreciable power is about 1.6x10^{18} Hz. It corresponds to photons of energy 6.6 keV from dipole magnets.

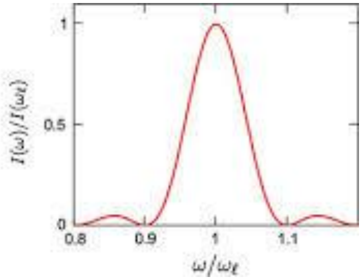


Enhancement of Synchrotron Radiation



radiation bandwidth $\frac{\Delta\omega}{\omega_0} \approx \frac{1}{N}$





$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\theta^2\right)$$

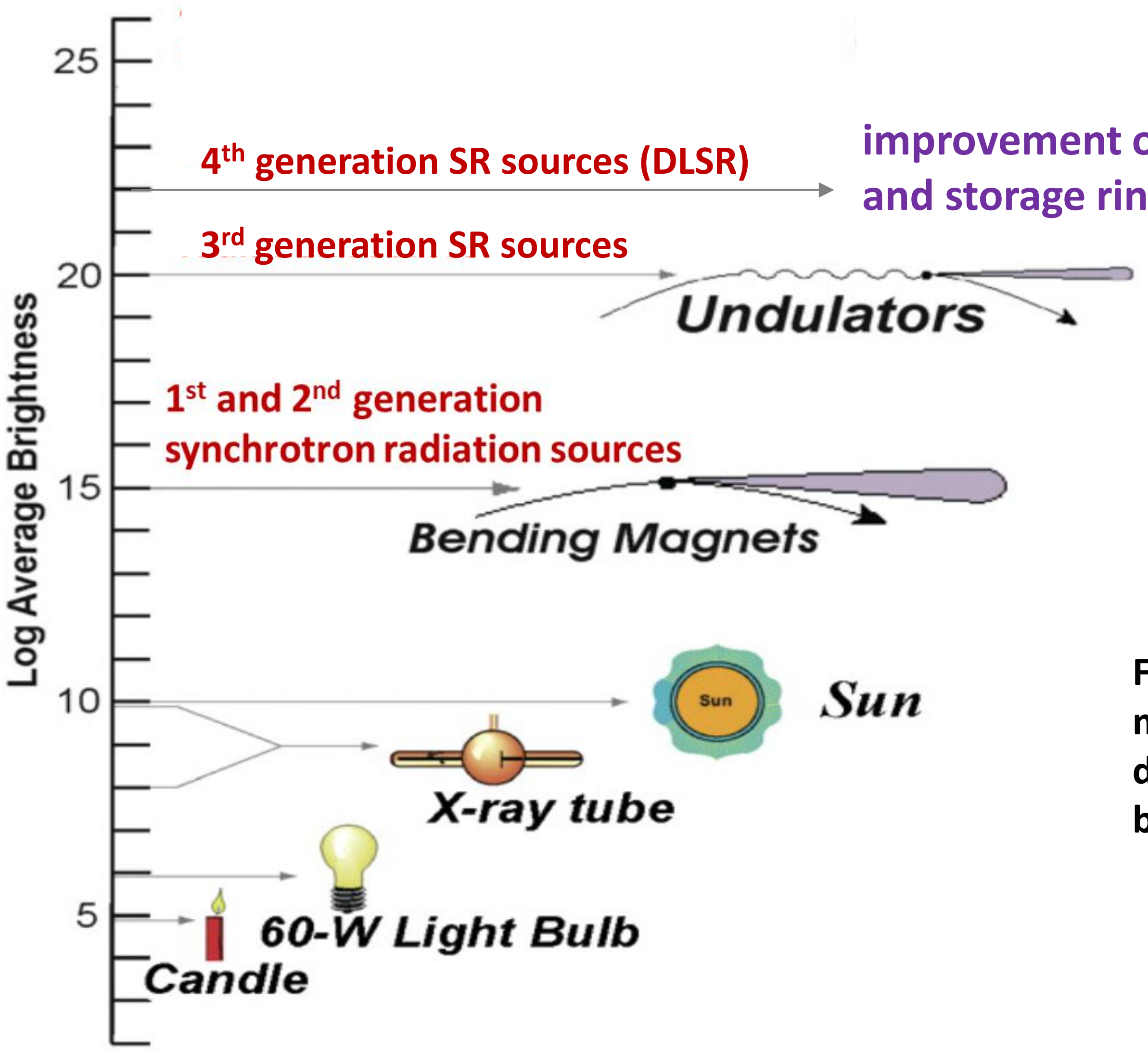
$$K = 0.934\lambda_u[cm]B_u[T]$$

emitted optical pulse
after N undulation cycle

$$\frac{\sin^2(N\pi u)}{(N\pi u)^2}$$

https://i.stack.imgur.com/jRAPD.png

Brightness of Synchrotron Light Sources



improvement of lattice design and storage ring technology

$$B = \frac{\dot{N}_{ph}}{4\pi^2 \Sigma_x \Sigma_y \Sigma_{x'} \Sigma_{y'} \frac{d\omega}{\omega}}$$

Brightness = # of photons/seconds/unit solid angle/0.1% BW

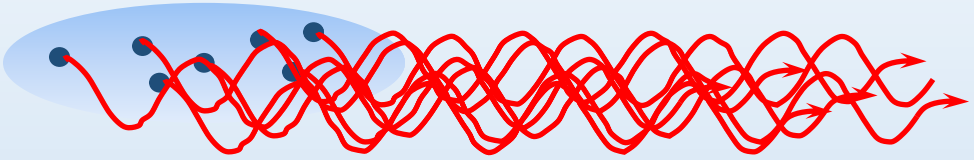
$$\Sigma_{x,y} = \sqrt{\sigma_{x,e}^2 + \sigma_{ph}^2}, \qquad \Sigma_{x,y} = \sqrt{\sigma_{x',e}^2 + \sigma'^2_{ph}}$$

For 3GLS, electron beam size $\sigma_{x,e}$ and divergence $\sigma'_{x,e}$ are much larger than single electron photon size σ_{ph} and divergence due to diffraction σ'_{ph} . The brightness can then be defined as

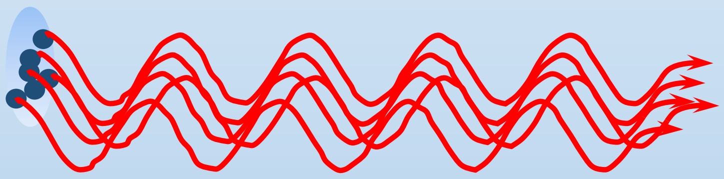
$$B \approx \frac{\dot{N}_{ph}}{4\pi^2 \varepsilon_x \varepsilon_y \frac{d\omega}{\omega}}$$

ε_x, ε_y are the horizontal and vertical beam emittances respectively.

Temporal Coherent Radiation by a Short Bunch



electrons radiate incoherently



electrons radiate coherently

$$= \sum_{k}^{N_e} \left| \vec{E}_k \right|^2 + \sum_{k \neq j}^{N_e} \vec{E}_k(\omega) \cdot \vec{E}_j(\omega)$$

Radiation field from a single electron (say the k[th] electron)

$$\vec{E}_k(\omega) = \vec{E}_0 e^{i(\omega t + \varphi_k)}$$

Radiation power from a bunch of electrons

$$P(\omega) \propto \left(\sum_{k}^{N_e} \vec{E}_k \right) \cdot \left(\sum_{j}^{N_e} \vec{E}_j^* \right) \propto \sum_{k,j}^{N_e} e^{i(\omega t + \varphi_k)} e^{-i(\omega t + \varphi_j)}$$

$$= \sum_{k,j}^{N_e} e^{i(\varphi_k - \varphi_j)} = N_e + \sum_{k \neq j}^{N_e} e^{i(\varphi_k - \varphi_j)} = N_e + N_e(N_e - 1) \left| \left\langle e^{i\varphi} \right\rangle \right|^2$$

$$P(\omega) = \left[N_e + N_e(N_e - 1) \left| \left\langle e^{i\varphi} \right\rangle \right|^2 \right] P_0(\omega)$$

$P_0(\omega)$ is the single electron radiation power

For a bunch of electrons with Gaussian distribution $G(z)$ which is characterized by *RMS* bunch length σ_z,

$$G(z) = \frac{N_e}{\sqrt{2\pi}\sigma} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

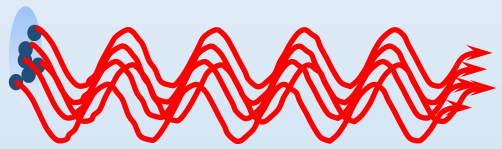
Then, the bunching factor of a beam with Gaussian distribution $g(\sigma)$ can be found as:

$$g(\sigma) \equiv \left|\left\langle e^{i\phi} \right\rangle\right| = \frac{\int G(z) e^{i\phi} dz}{\int G(z) dz}$$

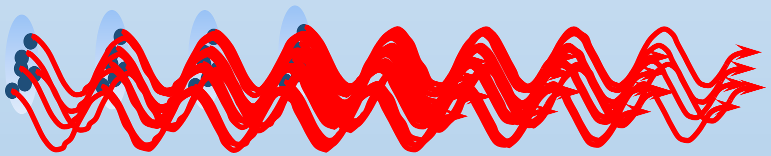
$$= \frac{\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2\sigma^2} + i\frac{2\pi z}{\lambda}\right) dz}{\int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2\sigma^2}\right) dz} = \exp\left(-2\pi^2\sigma^2/\lambda^2\right).$$

$$P(\omega) = \left[N_e + N_e(N_e - 1) g^2(\sigma)\right] P_0(\omega)$$

Temporal Coherent Radiation by Multiple Bunches



coherent radiation from a bunches of N_b electrons



coherent radiation from M bunches

Radiation power from a bunch of N_b electrons

$$P(\omega) \approx N_b^2 g^2(\sigma) P_0(\omega)$$

Radiation power from M bunches

$$P(\omega) \approx M^2 N_b^2 g^2(\sigma) P_0(\omega)$$

If we have a train of bunches moves 'coherently' in the undulator, line width of radiation is not limited by undulator length, but by total length of the bunch train. But how can we produce such bunch train??

Interaction of Electrons and EM Wave in Undulator

Consider an electron moving in a helical wiggler field,

$$\vec{B}_u = B_u \cos k_u z \hat{e}_x + B_u \sin k_u z \hat{e}_y$$

and interacting with a right-handed circular polarized wave:

$$\vec{E}_L = E_0 \cos \Phi \hat{e}_x - E_0 \sin \Phi \hat{e}_y$$

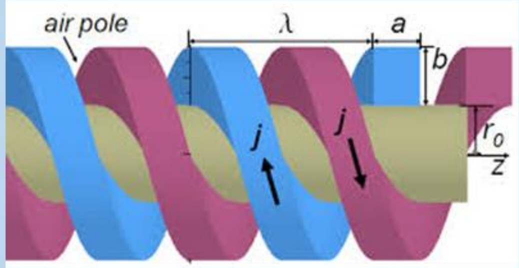
$$\vec{B}_L = \frac{E_0}{c} \sin \Phi \hat{e}_x + \frac{E_0}{c} \cos \Phi \hat{e}_y .$$

where $\Phi = kz - \omega t + \psi_0$, ψ_0 is the initial phase of the wave.

From Lorentz force equation,

$$\frac{d\vec{p}}{dt} = e\left[\vec{E}_L + \vec{v} \times \left(\vec{B}_u + \vec{B}_L\right)\right] \qquad \vec{p} = \gamma m \vec{v}$$

Helical undulator field can be generated by a *bifilar* helical current winding.



$$\vec{B}_u = 2B_u\left[I_1'(\lambda)\cos\chi\hat{e}_r - \frac{1}{\lambda}I_1(\lambda)\sin(\chi)\hat{e}_\theta + I_1(\lambda)\sin\chi\hat{e}_z\right]$$

where $\lambda = k_u r$, $\chi = \theta - k_u z$, I_1 and I_1' are the 1st order modified Bessel function of the first kind and its derivatives respectively. In the limit $r \ll \lambda_u$,

$$\vec{B}_u = B_u(\hat{x}\cos k_u z + \hat{y}\sin k_u z)$$

Consider an electron with initial velocity $v_z = v_0$, its transverse velocity in the undulator is:

$$\vec{v}_\perp = -\frac{eB_u}{mk_u\gamma}\left(\hat{x}\cos k_u z + \hat{y}\sin k_u z\right)$$

$$\vec{\beta}_\perp = -\frac{K}{\gamma}\left(\hat{x}\cos k_u z + \hat{y}\sin k_u z\right)$$

$$\because m\gamma\frac{d\vec{v}_\perp}{dt} = ev_z\hat{e}_z \times \vec{B}_u$$

$$\beta_z = \left[\beta^2 - \left(\frac{K}{\gamma}\right)^2\right]^{1/2}$$

electrons are moving at constant longitudinal velocities in helical undulators. However, this is not the case in planar undulators.

where

$$K = \frac{eB_u}{mck_u} = \frac{e\lambda_u B_u}{2\pi mc}$$

undulator parameter

$$K = 0.934\lambda_u[cm]B_u[T]$$

$$x' = -\frac{eA_u}{mc}\cdot\frac{c}{\gamma v_z}\cos k_u z = -\frac{K}{\gamma}\cdot\frac{1}{\beta_z}\cos k_u z$$

$$v_x = v_z x' = -\frac{eA_u}{m\gamma}\cos k_u z$$

$$v_y = v_z y' = -\frac{eA_u}{m\gamma}\sin k_u z$$

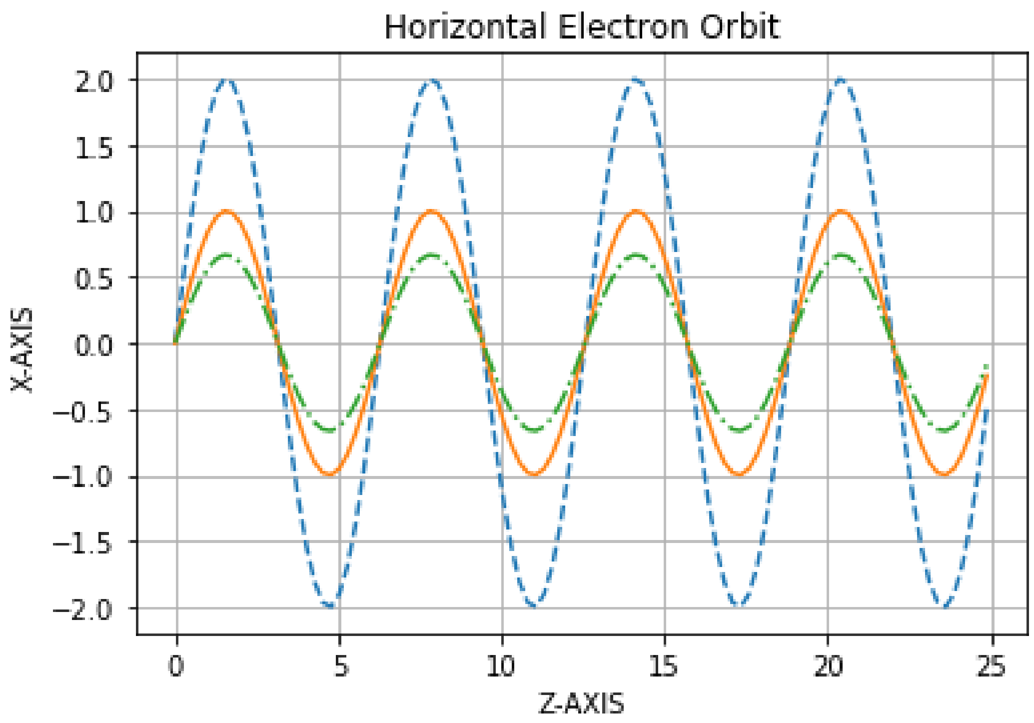
integrate

$v_z \approx v_0$

$$x = -\frac{eA_u}{m\gamma k_u v_0}\sin k_u z + x_0$$

$$y = \frac{eA_u}{m\gamma k_u v_0}\cos k_u z + y_0$$

electron orbit



Horizontal electron orbits of different energy:

- electrons of different γ are orbiting at the same period.
- maximum displacement from orbit center is inversely proportional to γ
- 'kick angle' is also inversely proportional to γ
- this is the origin of undulator dispersion

Electron dynamics in a laser field

Consider a right-hand circularly polarized light wave propagating along +z axis in the undulator field, then

$$\vec{E}_L = E_0 \cos\Phi \hat{x} - E_0 \sin\Phi \hat{y}$$

$$\vec{B}_L = \frac{E_0}{c} \sin\Phi \hat{x} + \frac{E_0}{c} \cos\Phi \hat{y}$$

$$\Phi = kz - \omega t + \psi_0$$

ψ_0 is the initial phase of the wave

with $\vec{p} = \gamma m \vec{v}$, from Lorentz force equation (in MKS units)

$$\frac{d\vec{p}}{dt} = -e\left[\vec{E}_L + \vec{v} \times \left(\vec{B}_u + \vec{B}_L\right)\right]$$

or

$$\frac{d\left(\gamma\vec{\beta}\right)}{dt} = -\frac{e}{mc}\left[\vec{E}_L + c\vec{\beta} \times \left(\vec{B}_u + \vec{B}_L\right)\right]$$

taking dot product with $\vec{\beta}$

$$\beta^2 \frac{d\gamma}{dt} + \gamma\vec{\beta}\cdot\frac{d\vec{\beta}}{dt} = -\frac{e}{mc}\vec{\beta}\cdot\left[\vec{E}_L + c\vec{\beta}\times\left(\vec{B}_u + \vec{B}_L\right)\right]$$

$$\left(1-\frac{1}{\gamma^2}\right)\frac{d\gamma}{dt}$$

$$\frac{\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}}{\left(1-\vec{\beta}\cdot\vec{\beta}\right)^{1/2}} = \frac{1}{\gamma^2}\frac{d\gamma}{dt}$$

$$\vec{\beta}_\perp\cdot\vec{E}_L$$

- laser field do not have longitudinal component
- magnetic forces do no works on the electron

define phase of the 'ponderomotive potential' as $\phi = (k+k_u)z-\omega t$

$$\frac{d\gamma}{dt} = -\frac{e}{mc}\left(\vec{\beta}_\perp\cdot\vec{E}_L\right) = \frac{eE_0}{mc}\cdot\frac{K}{\gamma}\left(\cos\Phi\cos k_u z - \sin\Phi\sin k_u z\right) = \frac{eE_0}{mc}\cdot\frac{K}{\gamma}\cos\left(\phi+\psi_0\right)$$

$$\frac{d\gamma}{dz} = \frac{eE_0}{mcv_z}\cdot\frac{K}{\gamma}\cos\left(\phi+\psi_0\right) \approx \frac{eE_0}{mc^2}\cdot\frac{K}{\gamma}\cos\left(\phi+\psi_0\right)$$

energy exchange between electron and wave (laser field) per unit time is a periodic function of ϕ !!

for v_z approx. equals to c (or transverse velocities are small enough)

Recall ponderomotive phase $\phi \equiv (k + k_u)z - \omega t$

taking derivative with respect to z:

$$\frac{d\phi}{dz} = k + k_u - \omega \frac{1}{v_z} = k + k_u - k\frac{1}{\beta_z}$$

but $\beta_z = \left[\beta^2 - \left(\frac{K}{\gamma}\right)^2\right]^{1/2} = \left[\left(1 - \frac{1}{\gamma^2}\right) - \left(\frac{K}{\gamma}\right)^2\right]^{1/2} \approx \left[1 - \left(\frac{1+K^2}{2\gamma^2}\right)\right]$

in the limit $\beta >> \beta_\perp$ or K/γ
and $\beta \rightarrow 1$

we have $\frac{d\phi}{dz} = k_u - k\left(\frac{1+K^2}{2\gamma^2}\right)$

'equation of phase advance'

Resonance Condition

If we choose $\gamma = \gamma_0$ such that no phase slippage between the particle and the ponderomotive wave (i.e. $d\phi/dz = 0$), then we have

$$k_u - k\left(\frac{1+K^2}{2\gamma_0^2}\right) = 0$$

in terms of wavelengths

$$\lambda = \frac{\lambda_u}{2\gamma_0^2}\left(1+K^2\right)$$

this is the so-called 'undulator equation' (i.e. the resonance condition) that predicts the central wavelength of spontaneous radiation from a helical undulator with undulator parameter *K* at a given electron energy.

If we define $\eta \equiv (\gamma - \gamma_0)/\gamma_0$ and assume $\eta << 1$

$$\Rightarrow \quad \frac{d\phi}{dz} = k_u - k\frac{1+K^2}{2\gamma_0^2}\cdot\frac{\gamma_0^2}{\gamma^2} = k_u\left(1-\frac{\gamma_0^2}{\gamma^2}\right) \approx 2k_u\eta$$

The Pendulum Equations

For $\eta << 1$

$$\frac{d\eta}{dz} = \frac{k a_L a_u}{\gamma_0^2}\cos(\phi + \psi_0)$$

$$\frac{d\phi}{dz} = 2k_u\eta$$

$$\frac{d^2\phi}{dz^2} = \frac{2kk_u a_L a_u}{\gamma_0^2}\cos(\phi + \psi_0)$$

nonlinear oscillation!!

constants

$$a_L = \frac{eA_L}{mc} = \frac{eE_0}{mc\omega_L} = \frac{eE_0}{mc^2k} \quad \text{and} \quad a_u = K$$

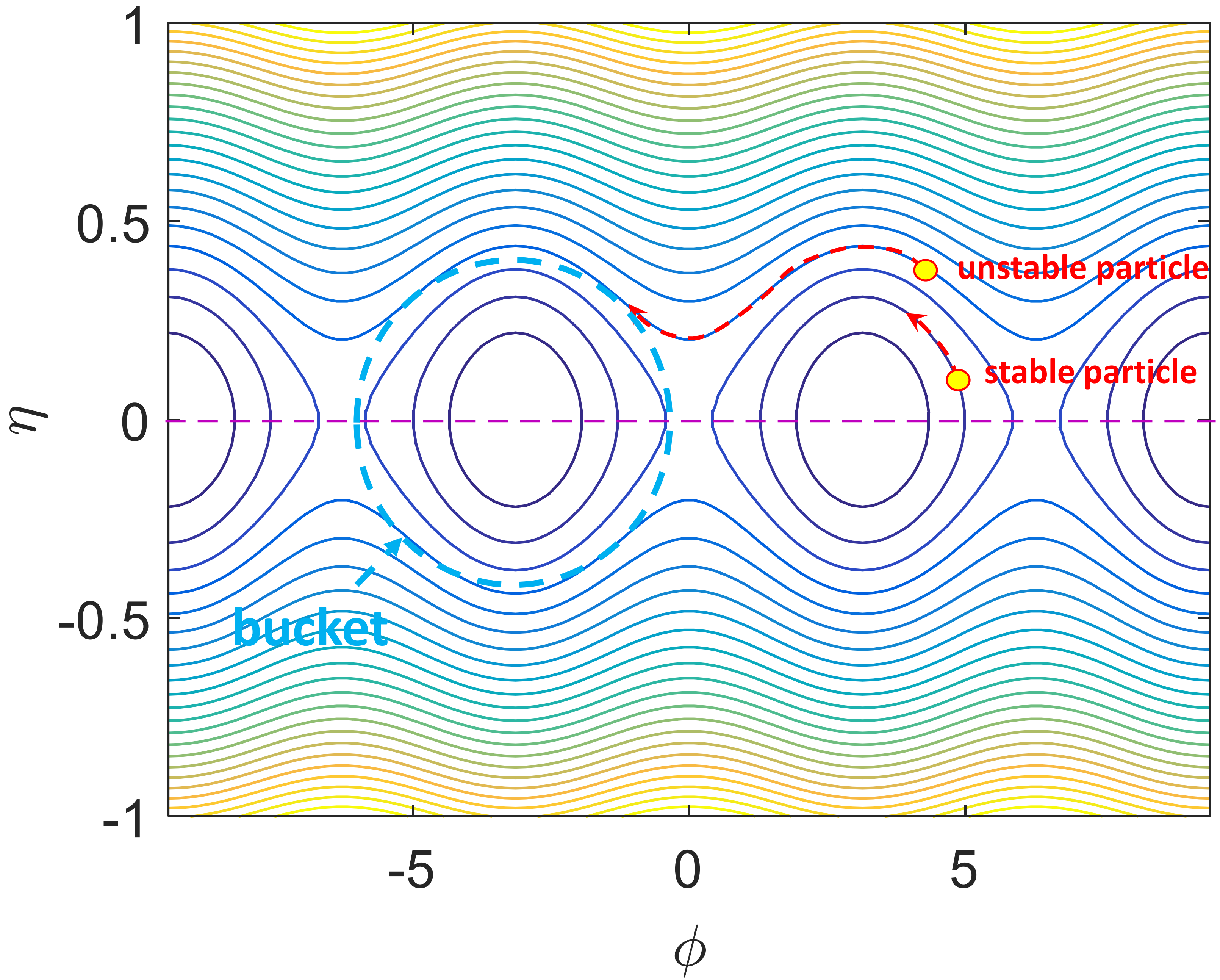
$$\because \frac{d\phi}{dz}\frac{d^2\phi}{dz^2} = \frac{1}{2}\frac{d}{dz}\left(\frac{d\phi}{dz}\right)^2$$

$$\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2 = \int \frac{d^2\phi}{dz^2}\cdot\frac{d\phi}{dz}dz = \int \frac{d^2\phi}{dz^2}d\phi + C$$

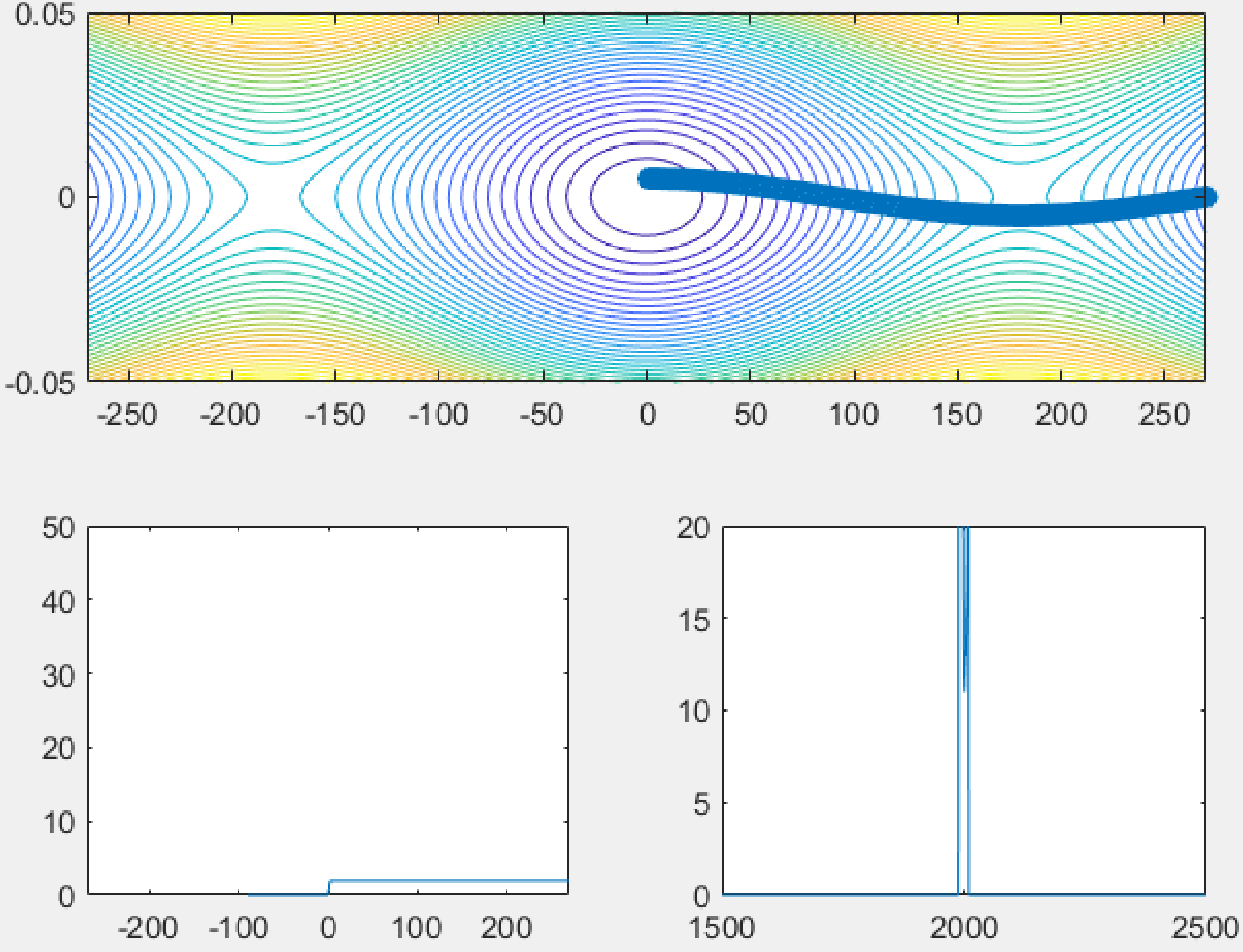
constant of integration

$$\frac{1}{2}\eta^2 + \frac{2kk_u a_L a_u}{\gamma_0^2}\sin(\phi + \psi_0) = C$$

'kinetic energy' 'potential energy'

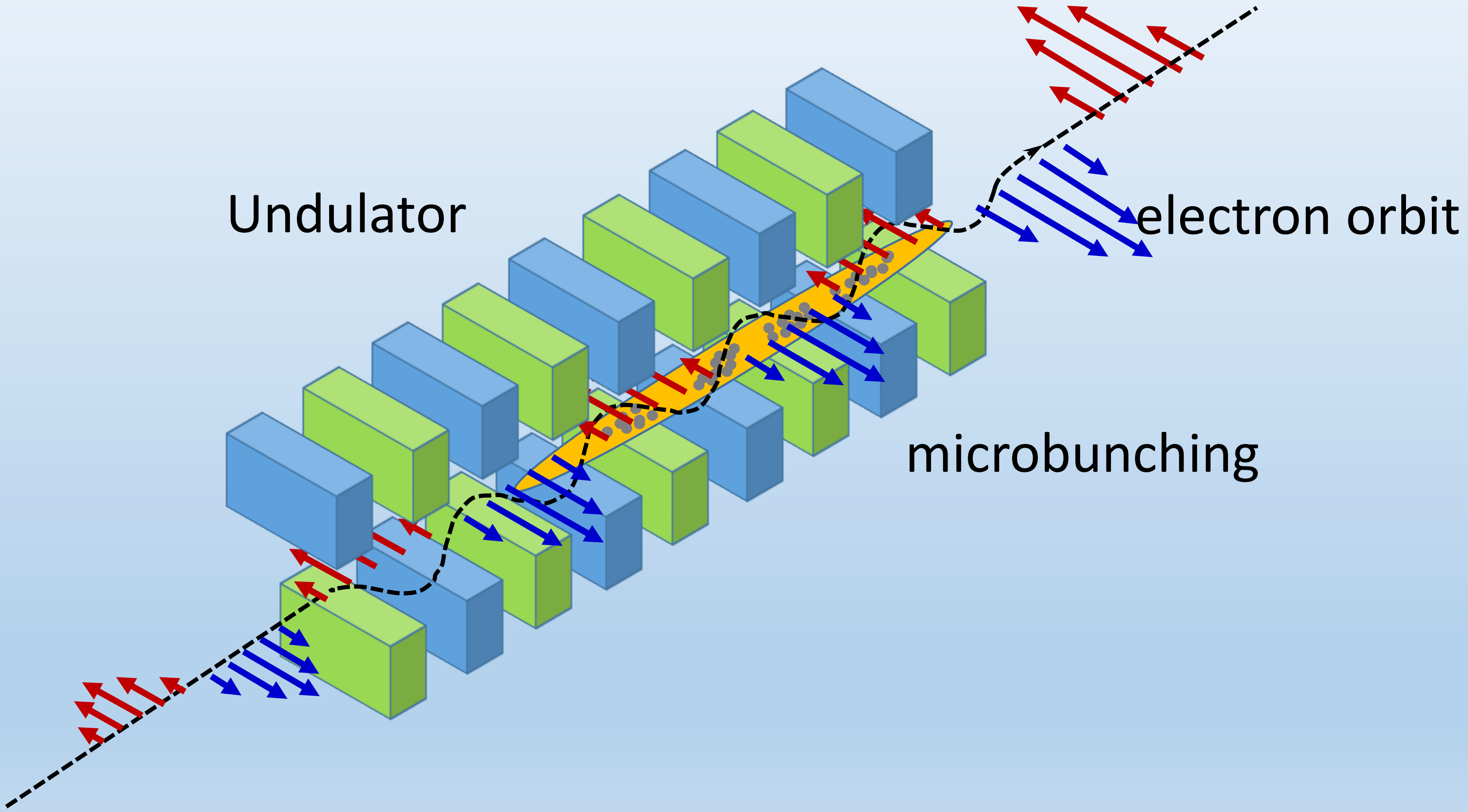


1
0.5
0
-0.5
-1
η
-5
0
5
ϕ
unstable particle
stable particle
bucket



0.05
0
-0.05
-250
-200
-150
-100
-50
0
50
100
150
200
250
50
40
30
20
10
0
-200
-100
0
100
200
20
15
10
5
0
1500
2000
2500

Beam-wave Interaction in Undulator



define FEL gain (small signal) *G* as:

$$G = -\frac{\langle \delta\gamma \rangle m_0 c^2 \times volume}{\varepsilon_0 E_0^2 \times volume}$$

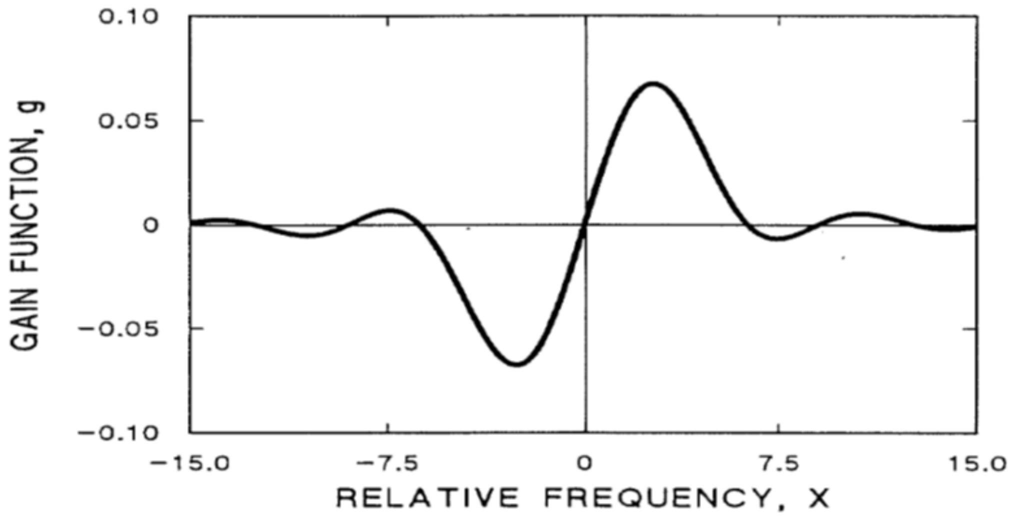
$$\approx -\frac{e^2 \rho_e}{2\varepsilon_0 m_0} \frac{a_L^2 a_u^2 \omega}{2\gamma_0^3 (\Delta\omega)^3} (1-\beta_{z0}) [2(1-\cos\Delta\omega t + \Delta\omega t \sin\Delta\omega t)]$$

let $$G_0 = \frac{e^2 \rho_e}{2\varepsilon_0 m_0} \frac{a_L^2 a_u^2 \omega (1-\beta_{z0}) L^3}{2\gamma_0^3 c^3 \beta_{z0}^3}$$ and $\eta = \Delta\omega L / v_{z0}$,

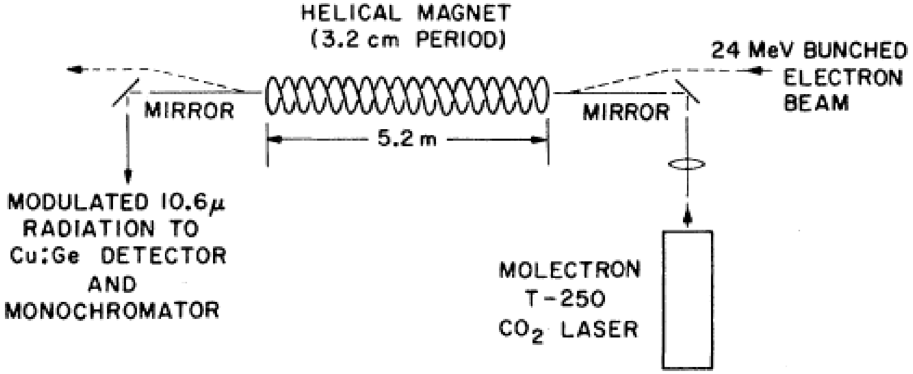
small signal FEL gain is given as:

$$G(\eta) = -G_0 g(\eta)$$

with $$g(\eta) \equiv [2(1-\cos\eta) - \eta \sin\eta] \frac{1}{\eta^3}$$



The First FEL Experiment



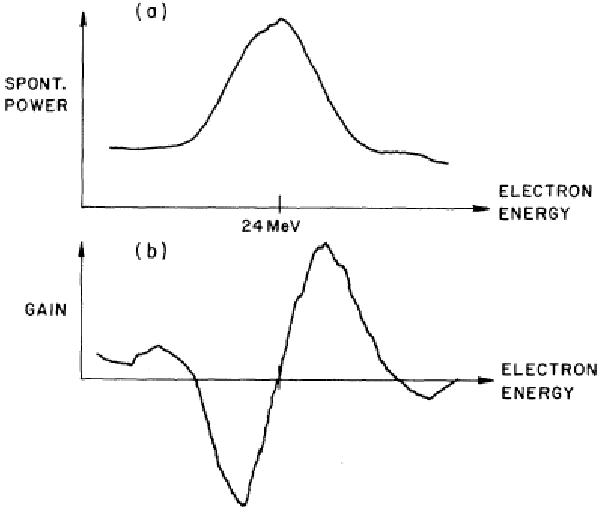
John Madey (1934 – 2016)

VOLUME 36, NUMBER 13 PHYSICAL REVIEW LETTERS 29 MARCH 1976

Observation of Stimulated Emission of Radiation by Relativistic Electrons in a Spatially Periodic Transverse Magnetic Field*

Luis R. Elias, William M. Fairbank, John M. J. Madey, H. Alan Schwettman, and Todd I. Smith
Department of Physics and High Energy Physics Laboratory, Stanford University, Stanford, California 94305
(Received 15 December 1975)

Gain has been observed for optical radiation at 10.6 μm due to stimulated radiation by a relativistic electron beam in a constant spatially periodic transverse magnetic field. A gain of 7% per pass was obtained at an electron current of 70 mA. The experiments indicate the possibility of a new class of tunable high-power free-electron lasers.



FEL Oscillator

the first working FEL!!

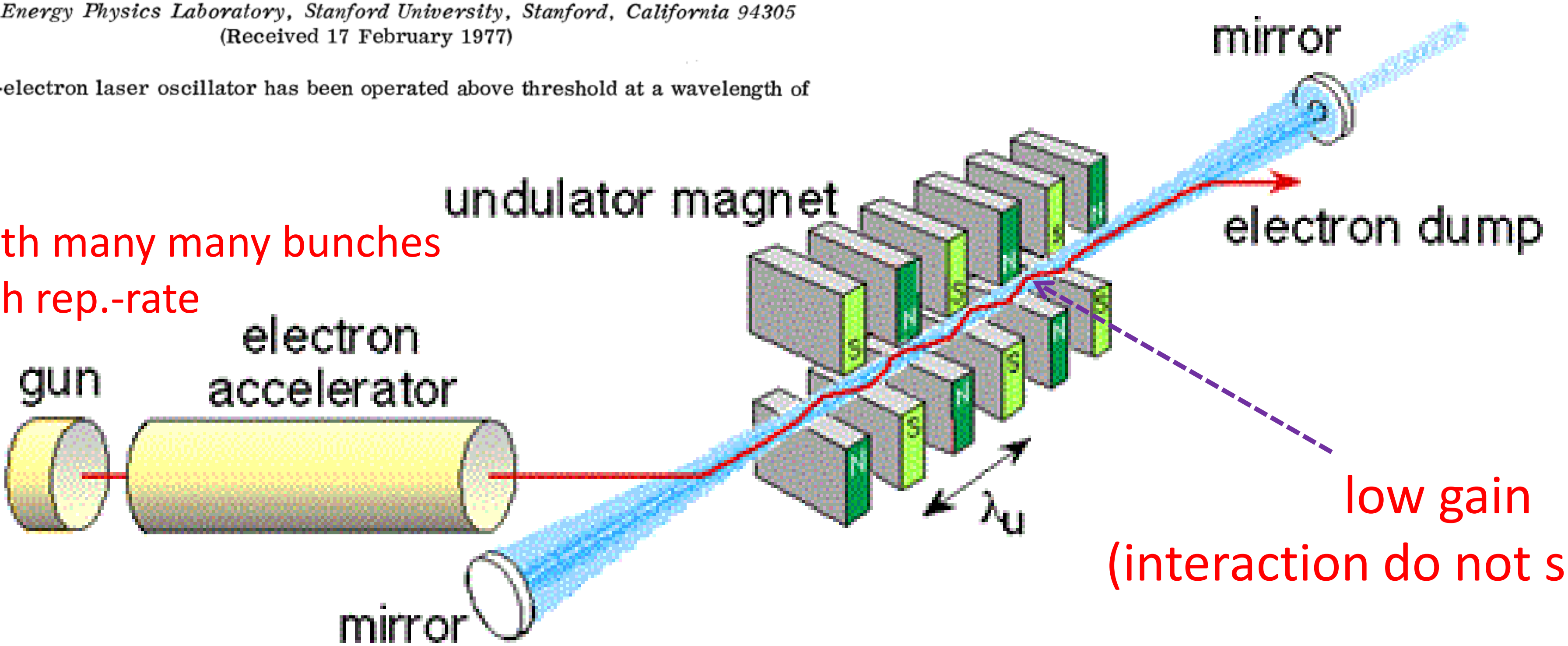
First Operation of a Free-Electron Laser*

D. A. G. Deacon,† L. R. Elias, J. M. J. Madey, G. J. Ramian, H. A. Schwettman, and T. I. Smith
High Energy Physics Laboratory, Stanford University, Stanford, California 94305
(Received 17 February 1977)

A free-electron laser oscillator has been operated above threshold at a wavelength of 3.4 μm.

(hole for output coupling)

drive beam with many many bunches
and high bunch rep.-rate



low gain
(interaction do not saturates)

optical cavity with high reflectivity mirrors
(no good mirrors < 200 nm)

Challenges for X-ray FELs

“Finite gain is available from the far-infrared through the visible region raising the possibility of continuously tunable amplifiers and oscillators at these frequencies with the *further possibility of partially coherent radiation sources in the ultraviolet and x-ray regions to beyond 10 keV*. Several numerical examples are considered.”

John M. J. Madey in “Stimulated Emission of Bremsstrahlung in a Periodic Magnetic Field”, Journal of Applied Physics Vol. 42, Number 1 (1971)

- No high reflectance mirrors in VUV and x-ray ranges
- Lack of seed lasers in beyond soft x-ray range
- To achieve high gain in a single pass, one has to have a quality electron beams at high peak current
- May need a long undulator.

- In Vlasov beam model, one has to solve the following equations self-consistently (Maxwell-Vlasov equations):

$$\frac{\partial f}{\partial t}+\sum_{i=1}^{3}\left[\frac{\partial f}{\partial q_i}\dot{q}_i+q\left(\vec{E}+\vec{v}\times\vec{B}\right)_i\frac{\partial f}{\partial P_i}\right]=0$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B}=\mu_0 q\int\vec{v}f\left(q_i,P_i,t\right)d^3P+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\cdot\vec{E}=\frac{q}{\varepsilon_0}\int f\left(q_i,P_i,t\right)d^3P$$

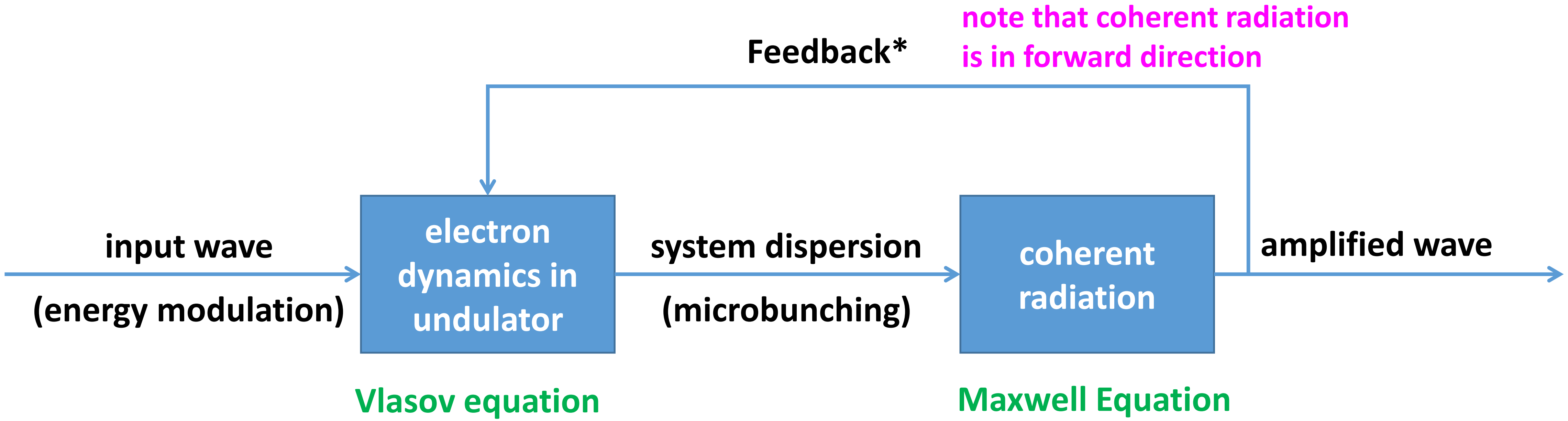
$$\nabla\cdot\vec{B}=0$$

- **Vlasov equation is, in general, nonlinear.**
- **given a initial beam distribution, integrate Vlasov numerically.**
- **determine an equilibrium state, linearize Vlasov equation w.r.t. this state and solve the linearized equation for small signals.**

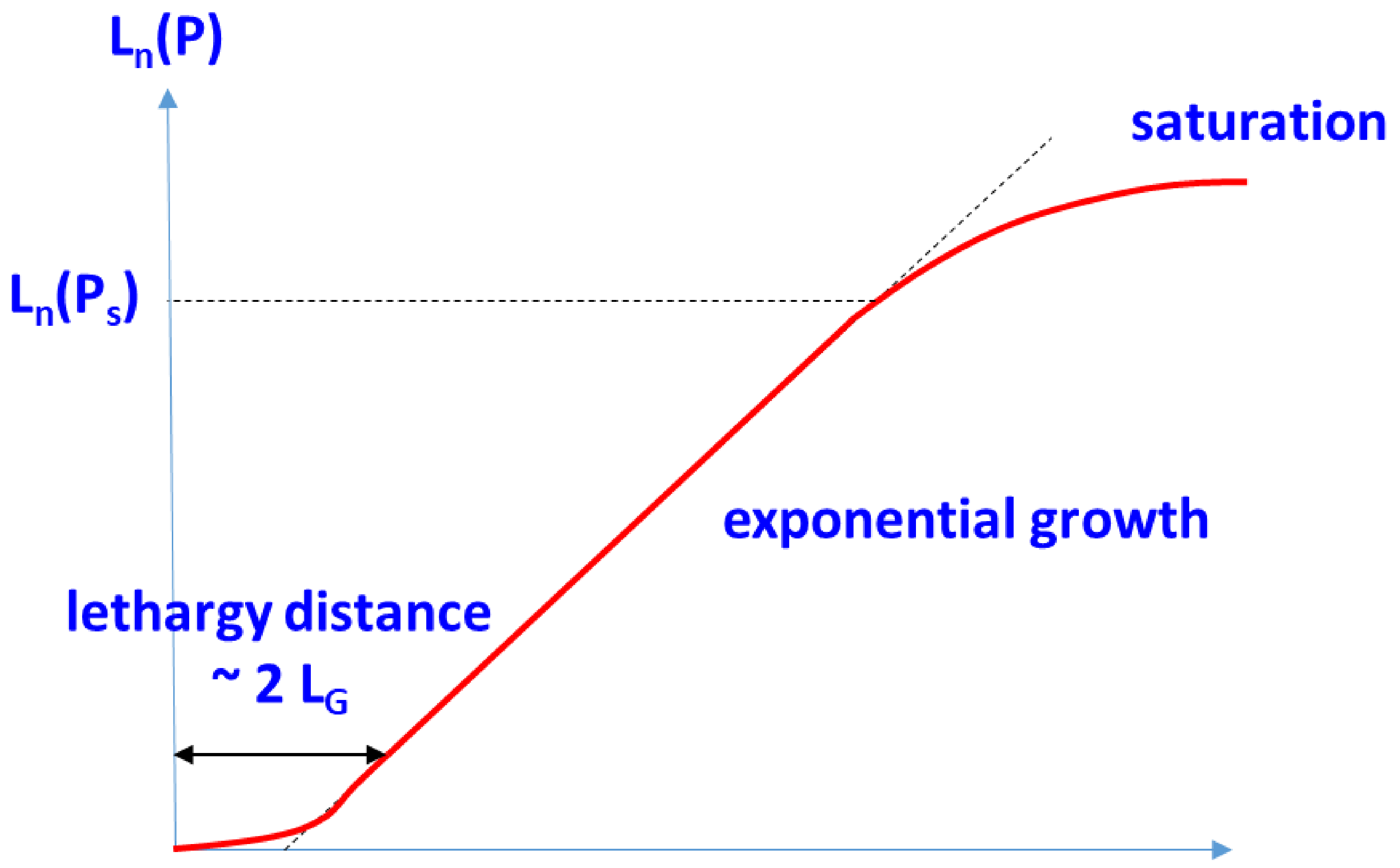
$$\vec{v}=\frac{\vec{P}}{\gamma m}=\frac{\vec{P}}{m}\left(1+\frac{P^2}{m^2c^2}\right)^{-1/2}$$

1D Model of Beam-Wave Interaction in Helical Undulator

- Neglecting the transverse variation of the radiation field.
- Assume the wiggler's gap and width are much larger than the beam size such that magnetic field is approx. constant within the beam size.
- Beam-wave interaction is strong enough, electron dynamics in an undulator is affected by the radiation field and if a positive feedback mechanism has been setup, the amplitude of the radiation field grows exponentially.



Evolution of Radiation Power in FEL



Major Performance Parameters for High Gain FELs

For a FEL amplifier, the growth and saturation of radiation can be described by:

$$P(z) = \alpha P_n e^{z/L_g} < P_{sat}$$

α is the coupling coefficient, P_n is the input power. For SASE, the input noise power is the frequency integrated synchrotron radiation power in an FEL gain bandwidth generated in the first gain length. L_g is the gain length, L_{sat} is the saturation length, P_{sat} is the saturated power. The saturation length is given by

$$L_{sat} = L_g \ln\left(\frac{P_{sat}}{\alpha P_n}\right)$$

L_g , L_{sat} and P_{sat} are the major performance parameters for a high gain FEL amplifier.

Formulas for 1D SASE FEL Theory

- Coupling coefficient, α:

$$\alpha = 1/9$$

- Effective input noise power:

$$P_n \approx \rho^2 c E_0 / \lambda$$

- 1D gain length:

$$L_{1D} = \lambda_u / 4\pi\sqrt{3}\rho$$

1D model gives the highest possible FEL gain (shortest gain length).

Formulas for 1D SASE FEL Theory (cont'd)

- Saturation power P_{sat}:

$$P_{sat} \approx \rho P_{beam}$$

- Pierce parameter ρ:

$$\rho = \left[\left(\frac{I}{I_A}\right)\left(\frac{\lambda_u A_u}{2\pi\sigma_x}\right)^2\left(\frac{1}{2\gamma_0}\right)^3\right]^{1/3}$$

I_A = 17.045 kA is the Alfven current,

$A_u = a_u$ for helical undulator

$A_u = a_u\left[J_0(\xi) - J_1(\xi)\right]$ for planar undulator.

$\xi = a_u^2 / 2\left(1 + a_u^2\right)$.

$\sigma_x = \sqrt{\beta\,\varepsilon_n/\gamma_0}$ is the electron beam size.

$P_{beam}[TW] = E_0[GeV]I[kA]$ is the beam power.

Beam Quality Requirements of High Gain FELs

Acceptable beam emittance is defined by the relation:

$$\varepsilon \leq \frac{\lambda_{\mathrm{FEL}}}{4\pi}$$

Energy spread criteria:

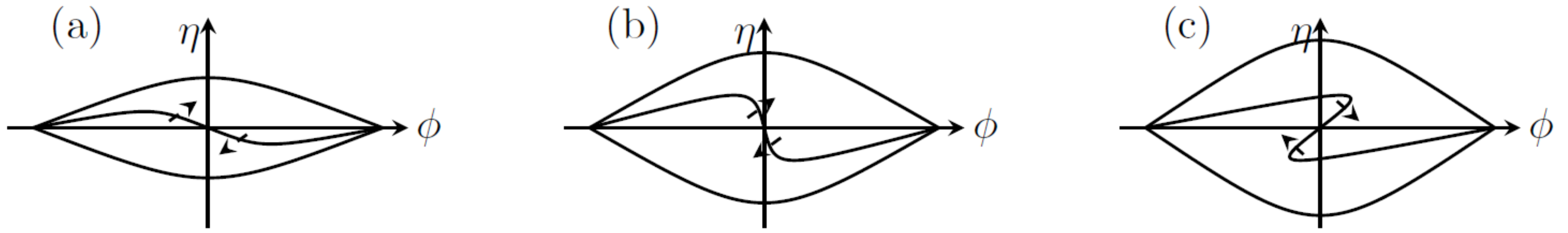
$$\frac{\Delta E}{E_0} < \rho$$

electron density

$$\rho = \frac{1}{2}\left[Z_0 \cdot \frac{e n_0 c}{k_w^2} \cdot \frac{e}{mc^2} \cdot \frac{K^2 [JJ]^2}{4\gamma_0^3} \right]^{1/3}$$

Pierce parameter

FEL Saturation



- The electrons oscillates in phase space at synchrotron frequency $\Omega_s^2 = 2D_2E/\gamma_0^2$. As the radiation field E grows exponentially, the *bucket height* in the phase space increases, and the *energy spread* of the beam also increases due to the interaction with the radiation field.
- As the radiation power increases, the electron distribution rotates faster and faster in the bucket (i.e. $\Omega_s \propto \sqrt{E}$) , but the growth rate of the field remains nearly the same.
- As a results, when the rotation is faster than growth rate and the rotation reaches near 90 degree in the bucket. The electrons can not radiate energy any more and start to absorb energy from the field. The FEL is said to reach its saturation.

Saturation power can be estimated when the synchrotron frequency Ω_s increases to be equal to the growth rate, it is found to be:

$$E_s = \frac{3\rho^2\gamma_0^2}{2D_2}$$

Power density for a helical wiggler is:

$$\frac{|E_s|^2}{Z_0} = \frac{9}{4}(\rho\gamma_0)^4 \frac{1}{Z_0 D_2^2} = \frac{9}{16}\rho n_0 cmc^2\gamma_0$$

Saturation power is:

$$P_s = \frac{|E_s|^2}{Z_0} A = \frac{9}{16}\rho n_0 cAmc^2\gamma_0$$

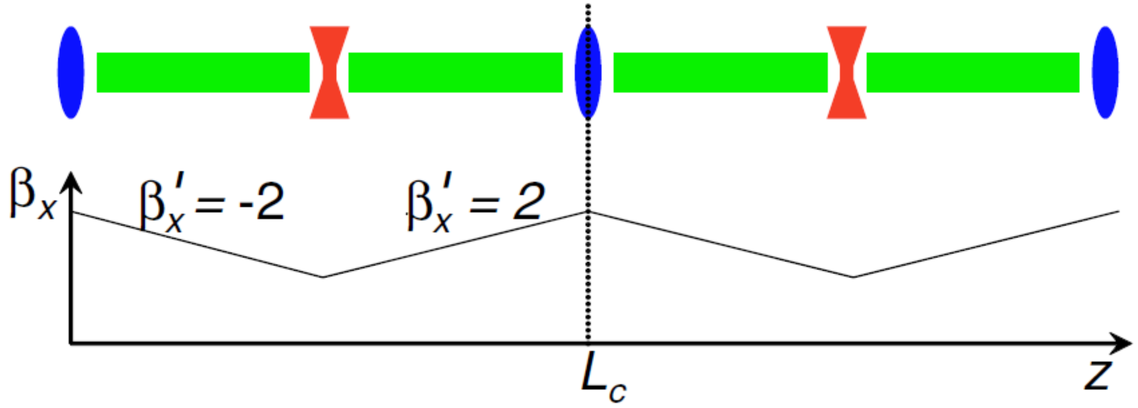
But n_0cA is the number of electrons per second, $n_0cAmc^2\gamma_0$ is the electron beam power P_e. We have:

$$\boxed{P_s \approx \rho P_e}$$

This is an important result because it implies ρ is the approximate FEL interaction efficiency!!

3D effects

- A beam with finite transverse emittance will have certain angular spread that makes the beam expands in size as it propagates along the undulator.
- Planar undulator will have natural focusing force.
- Strong focusing is usually used to keep the beam size nearly constant for effective FEL interaction
- Diffraction of radiation field has to be considered.



strong focusing of electron beam for long distance propagation

Xie's Fitting Formula

- The FEL gain length in 3D model can be expressed by a universal scaling function

$$\frac{L_{1D}}{L_g} = F\left(\eta_d, \eta_\varepsilon, \eta_\gamma\right)$$

where $\eta_d = \frac{L_{1D}}{L_R}, \eta_c = \left(\frac{L_{1D}}{\beta}\right)\left(\frac{4\pi\varepsilon}{\lambda}\right), \eta_\gamma = 4\pi\left(\frac{L_{1D}}{\lambda_u}\right)\left(\frac{\sigma_E}{E_0}\right)$ and $L_R = 4\pi\sigma_x^2 / \lambda$

is the Rayleigh range. The scaling parameters η_d, η_ε, and η_γ measure the non-ideal beam (the deviation of the beam from the ideal case). η_d is for the gain reduction due to diffraction, η_ε is for the gain reduction due to electron's longitudinal velocity spread caused by emittance, η_γ is for the gain reduction due to electron's longitudinal velocity spread caused by energy spread. L_g approaches L_{1D} if all the scaling parameters vanish.

Xie's Fitting Formula (cont'd)

- The universal scaling function is determined by fitting numerical solutions of the coupled Maxwell-Vlasov equations describing FEL interaction, is given by

$$\frac{L_{1D}}{L_g} = \frac{1}{1+\eta}$$

where

$$\eta = a_1\eta_d^{a_2} + a_3\eta_\varepsilon^{a_4} + a_5\eta_\gamma^{a_6}$$
$$+ a_7\eta_\varepsilon^{a_8}\eta_\gamma^{a_9} + a_{10}\eta_d^{a_{11}}\eta_\gamma^{a_{12}} + a_{13}\eta_d^{a_{14}}\eta_\varepsilon^{a_{15}}$$
$$+ a_{16}\eta_d^{a_{17}}\eta_\varepsilon^{a_{18}}\eta_\gamma^{a_{19}}$$

Xie's Fitting Parameters			
a1 = 0.45	a2 = 0.57	a3 = 0.55	a4 = 1.6
a5 = 3	a6 = 2	a7 = 0.35	a8 = 2.9
a9 = 2.4	a10 = 51	a11 = 0.95	a12 = 3
a13 = 5.4	a14 = 0.7	a15 = 1.9	a16 = 1140
a17 = 2.2	a18 = 2.9	a19 = 3.2	

- The saturation power obtained empirically by fitting simulation results is given by

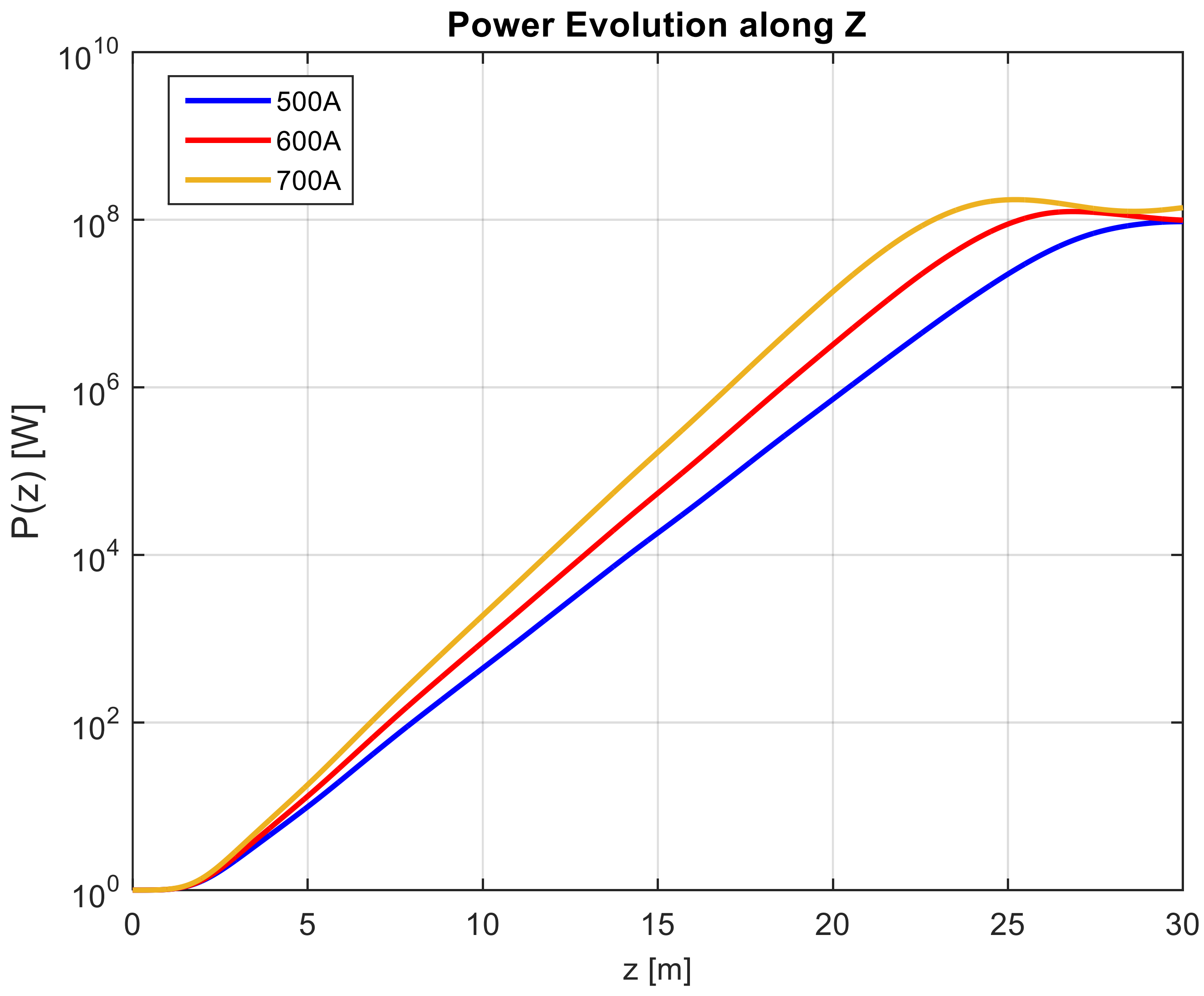
$$P_{sat} \approx 1.6\rho\left(\frac{L_{1D}}{L_g}\right)^2 P_{beam}$$

- A formula for noise power is not available for non-ideal beam

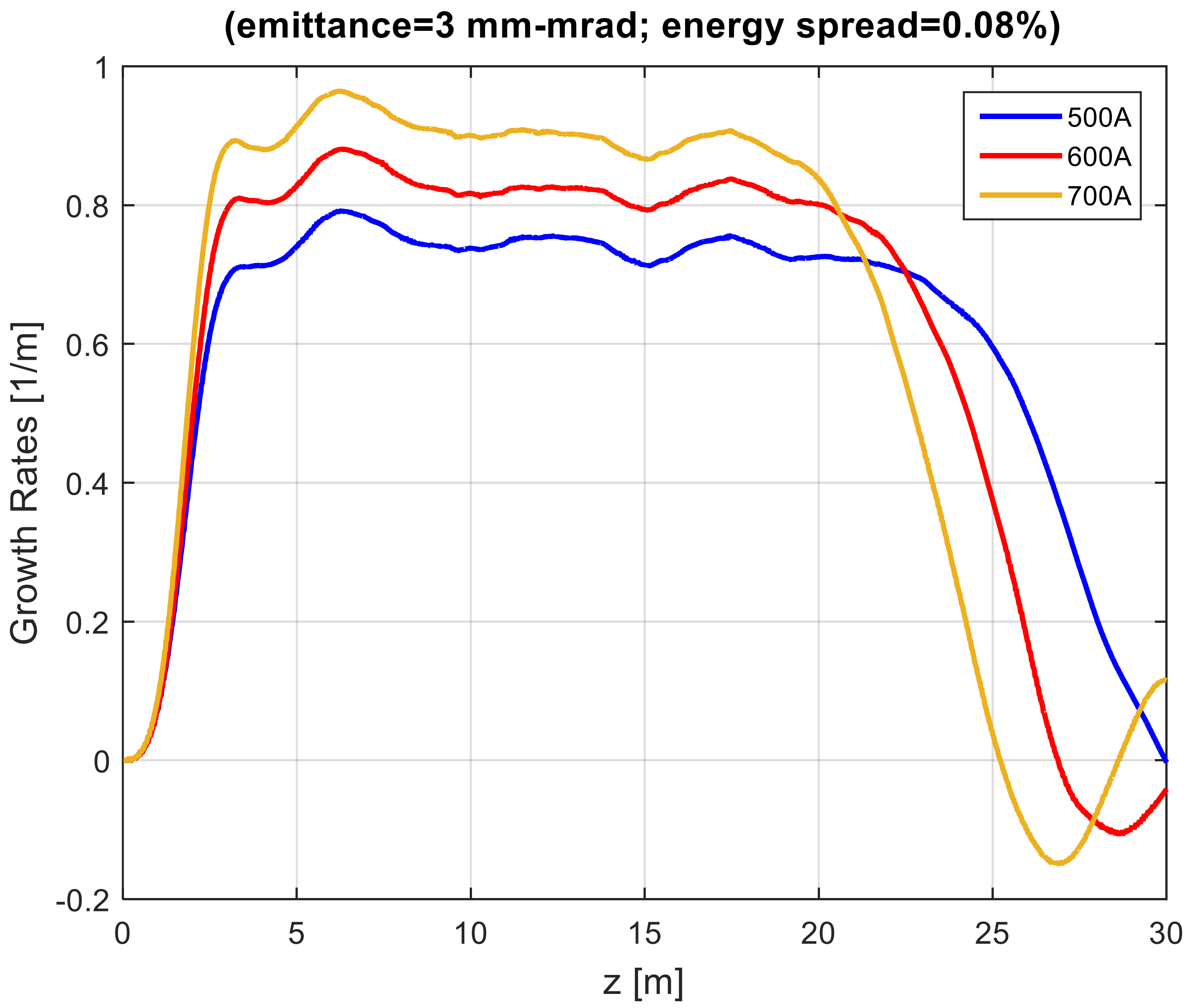
Input Parameters being used in this study

50-nm High Gain FEL Simulation Parameters	
Radiator period [mm]	20
Radiator type	helical
Radiator parameter, K	0.445
Operating field, B_0 [T]	0.238
Radiation wavelength [nm]	49.963
Electron beam energy [MeV]	250
β function [m]	4.95
Beam size, rms [μm]	174
Normalized emittance [mm-mrad]	3.0
Peak current [A]	500
Energy spread [%]	0.08
Input power [W]	1
Rayleigh range [m]	10.55

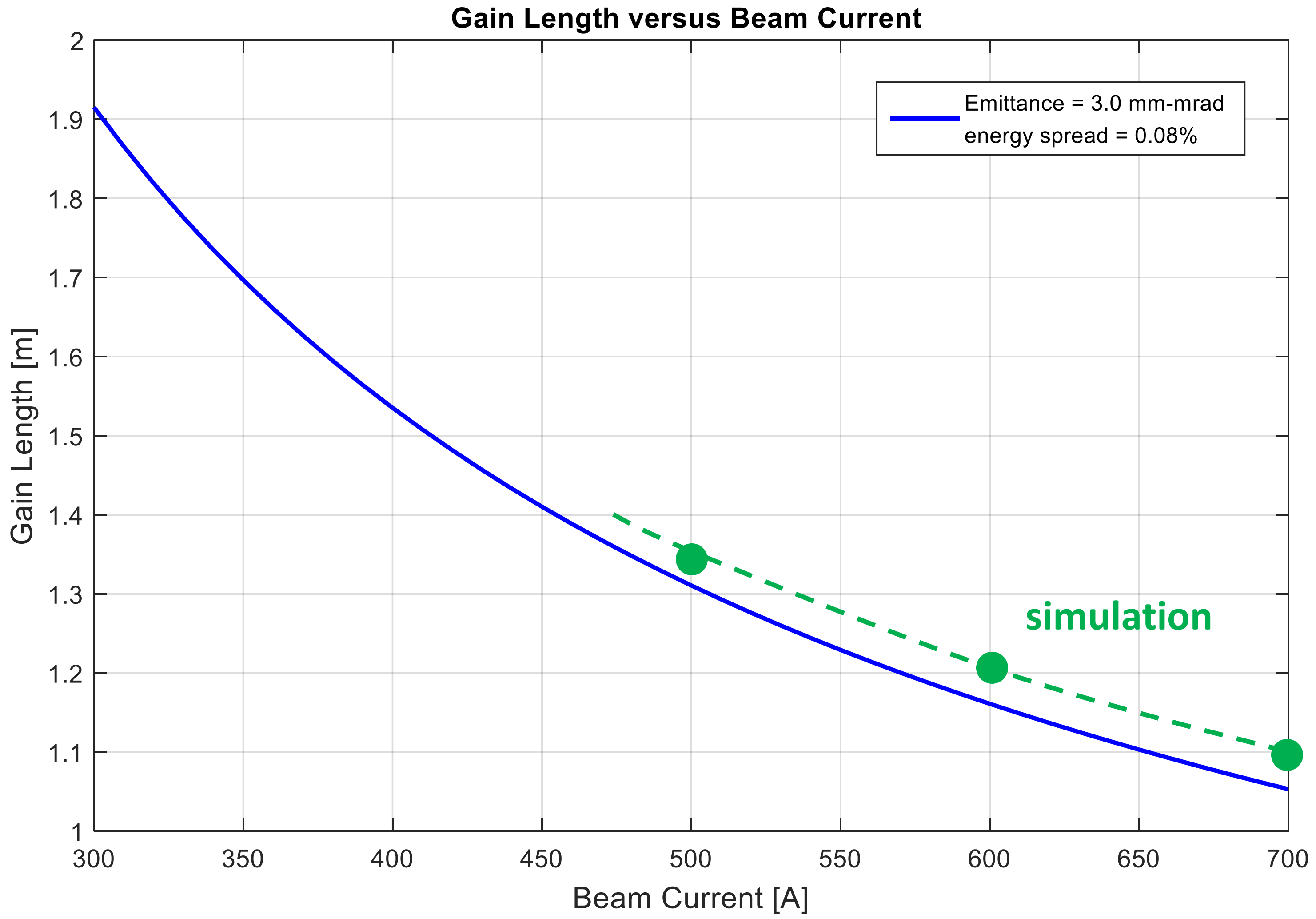
Power Evolution along Z



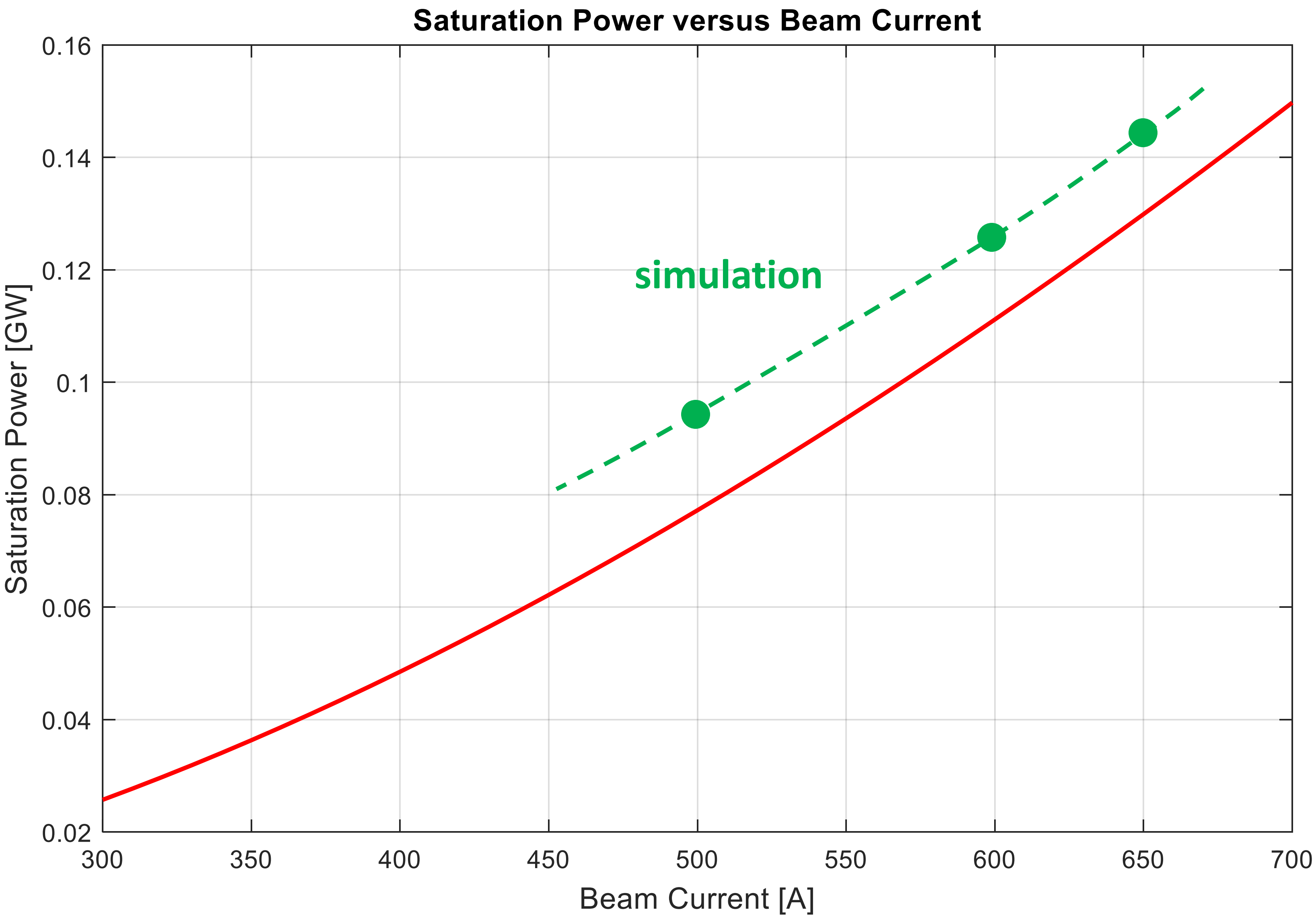
Evolution of Instantaneous Growth Rate



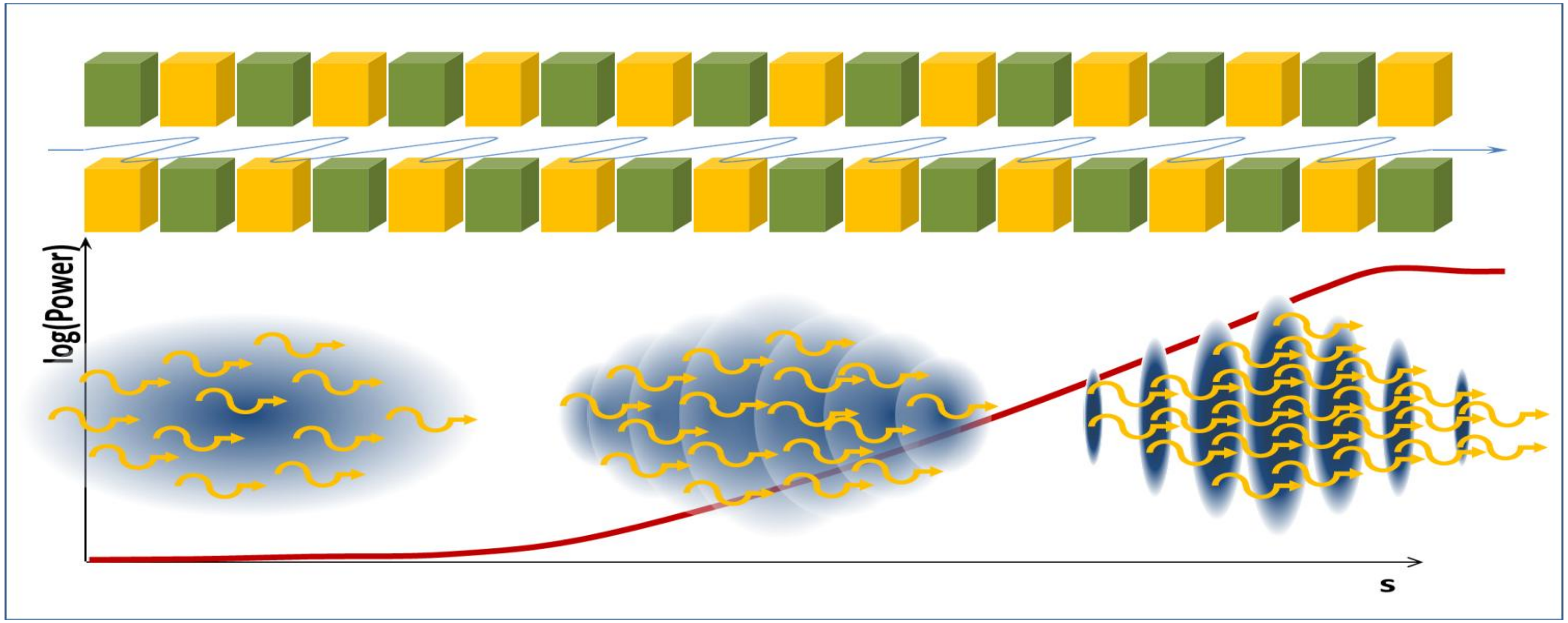
Simulation Results

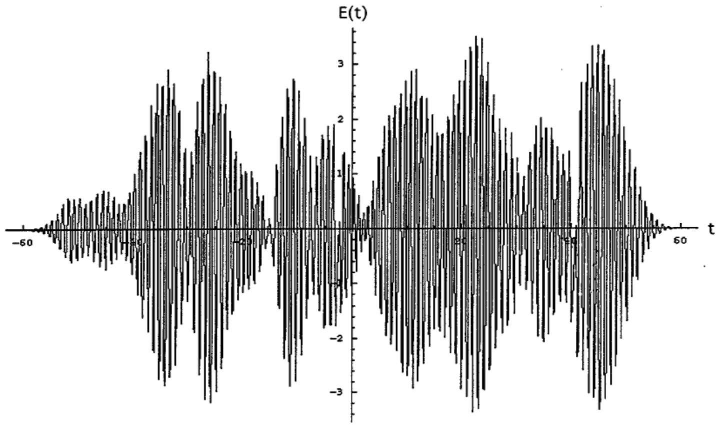


Simulation Results

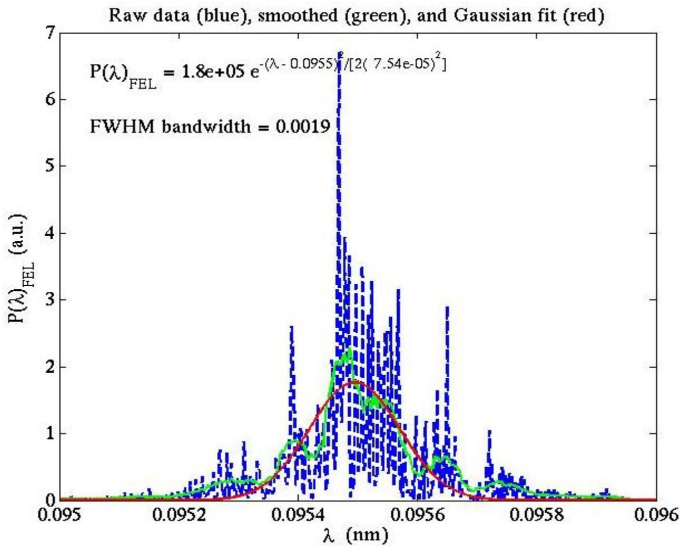


Self-amplification of Spontaneous Emission (SASE)

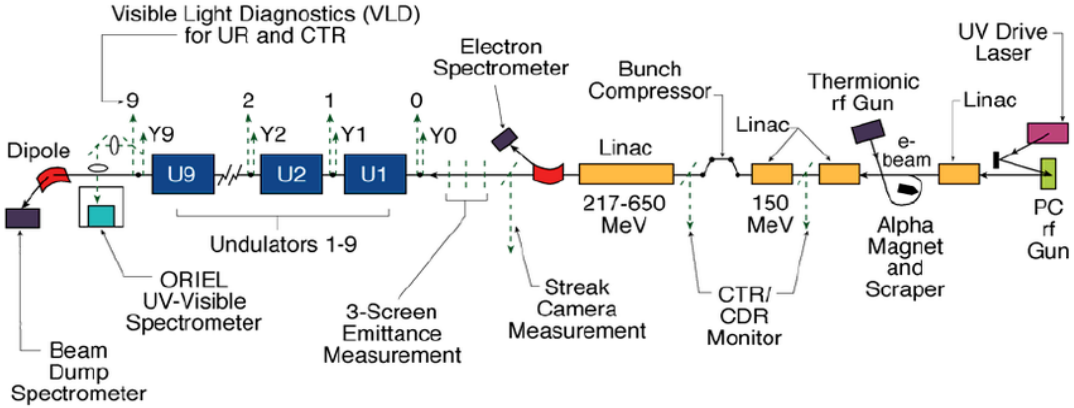


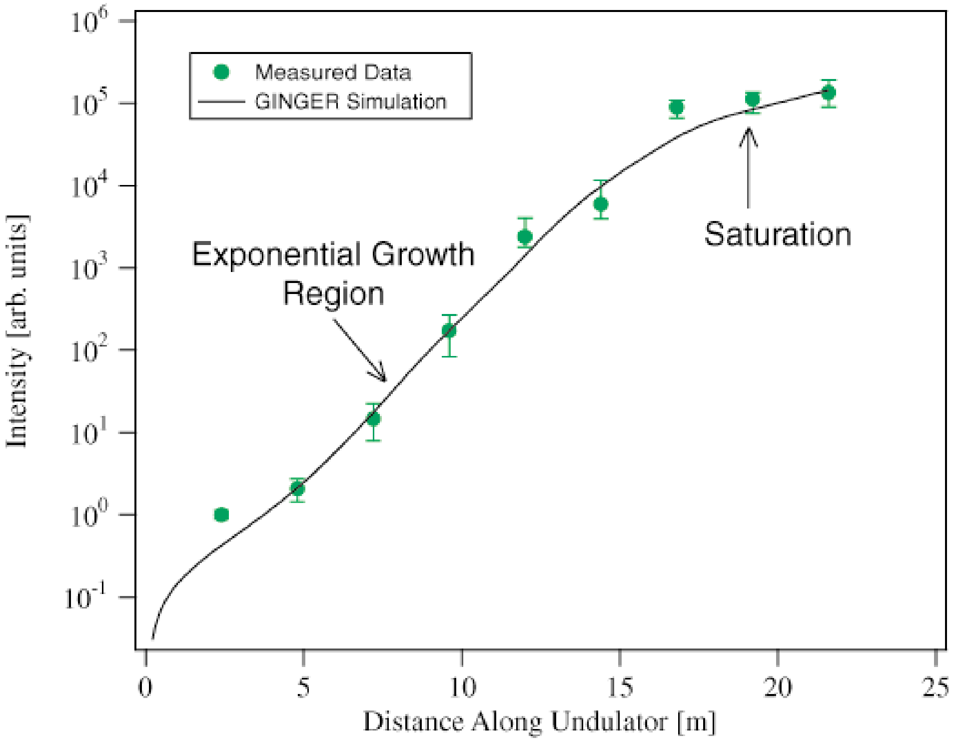


Example of LCLS-II 13 keV case



First SASE FEL @ Argonne National Lab





Sciencexpress Research Article

Exponential Gain and Saturation of a Self-Amplified Spontaneous Emission Free-Electron Laser

S. V. Milton,[1]* E. Gluskin,[1] N. D. Arnold,[1] C. Benson,[1] W. Berg,[1] S. G. Biedron,[1,2] M. Borland,[1] Y.-C. Chae,[1] R. J. Dejus,[1] P. K. Den Hartog,[1] B. Deriy,[1] M. Erdmann,[1] Y. I. Eidelman,[1] M. W. Hahne,[1] Z. Huang,[1] K.-J. Kim,[1] J. W. Lewellen,[1] Y. Li,[1] A. H. Lumpkin,[1] O. Makarov,[1] E. R. Moog,[1] A. Nassiri,[1] V. Sajaev,[1] R. Soliday,[1] B. J. Tieman,[1] E. M. Trakhtenberg,[1] G. Travish,[1] I. B. Vasserman,[1] N. A. Vinokurov,[3] G. Wiemerslage,[1] B. X. Yang[1]

[1]Advanced Photon Source, Argonne National Laboratory, Argonne, IL 60439, USA. [2]MAX-Laboratory, University of Lund, 221 00 Lund, Sweden. [3]Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russian Federation.

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Self-amplified spontaneous emission in a free-electron laser is a proposed technique for the generation of very high-brightness coherent x-rays. The process involves passing a high-energy, high-charge, short-pulse, low-energy-spread, and low-emittance electron beam through the periodic magnetic field of a long series of high-quality undulator magnets. The radiation produced grows exponentially in intensity until it reaches a saturation point. We report on the demonstration of self-amplified spontaneous emission gain, exponential growth, and saturation at wavelengths in the visible (530 nm) and ultraviolet (385 nm). Good agreement between theory and simulation indicates that scaling to much shorter wavelengths may be possible. These results confirm the physics behind the self-amplified spontaneous emission process and move us a step closer toward an operational x-ray free-electron laser.

Generation of high-brightness (photon flux per frequency bandwidth per unit phase space volume), hard x-rays (photon energies greater than roughly 5 keV or wavelengths less than 2.5 Å) has long been the domain of high-energy electron storage-ring-based, synchrotron light sources (*1–3*). However, significant advances in x-ray brightness could potentially be achieved using free-electron lasing action at these short wavelengths. Unfortunately, free-electron lasers (FELs) based on the oscillator principle and conventional laser systems are limited on the short wavelength side to ultraviolet wavelengths, primarily due to mirror or seed beam limitations.

A way to achieve free-electron lasing at wavelengths shorter than ultraviolet and including hard x-rays is known as a single-pass, high-gain FEL based on the self-amplified spontaneous emission process (SASE) (*4–8*). A high-quality, high-peak-current electron beam is accelerated and passed through an undulator (a long, high-quality, sinusoidally-varying magnetic field). A favorable instability begins between the electron beam and the electromagnetic (EM) wave it is producing, and the optical power increases exponentially until the process eventually saturates at some maximum radiation output level. At x-ray wavelengths the peak brightness would be much higher (by more than ten orders of magnitude) than the brightness of sources available today at comparable wavelengths.

Another significant feature is that this process can occur at any wavelength as it scales with the electron beam energy and so is continuously tunable in wavelength. Achieving saturation of the process is thus a matter of providing an undulator of sufficient length and quality and then passing a sufficiently high-energy, high-quality beam through the undulator field.

Previous measurements of the SASE process operating to saturation have been made, however none at wavelengths shorter than 585×10^3 nm (*9*). As a direct result of advances in the areas of high-brightness electron beam production using photocathode rf electron guns (*10, 11*) and long, high-quality undulator magnets such as those now used at all major synchrotron light source facilities, recent progress has been made at extending the measurements of the SASE process to shorter wavelengths (*12–14*), and in one case to a wavelength of 80 nm (*15*).

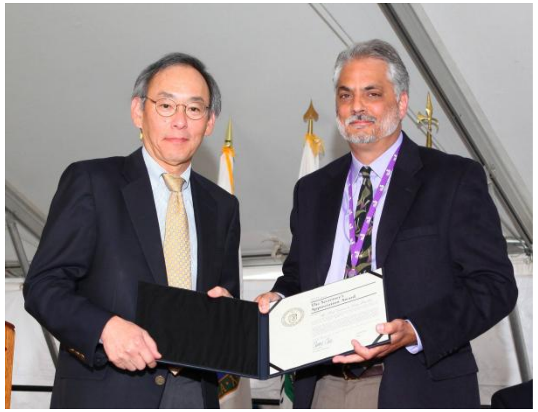
Our low-energy undulator test line (LEUTL) (Fig. 1) and its various component systems (*16–23*) are designed to achieve and explore the SASE FEL process to saturation in the visible and ultraviolet wavelengths and to explore topics of interest for a next-generation linac-based light source. Using a frequency-quadrupled Nd:Glass drive laser, high-quality electron bunches are generated via the photoelectric effect within a photocathode rf gun using copper as the cathode material. The electron bunch is initially accelerated to roughly 5 MeV, and is then injected into the linear accelerator and further accelerated to the desired energy (up to a maximum of 650 MeV). In addition to acceleration, the beam undergoes magnetic bunch compression to increase the peak current. Finally it is passed through the undulator field where SASE begins.

The essence of SASE lies in the generation of EM radiation by the electrons as they are transversely accelerated by the magnetic field of the undulator magnet and by the interaction of the EM field back on the electrons. When an electron beam traverses an undulator, it emits EM radiation at the resonant wavelength $\lambda_r = (\lambda_u/2\gamma^2)(1 + K^2/2)$. Here λ_u is the undulator period, γmc^2 is the electron beam energy, $K = eB_o\lambda_u/2\pi mc^2$ is the dimensionless undulator strength parameter, and B_o is the maximum on-axis magnetic field strength of the undulator. Although the EM wave is always faster than the electrons, a resonant condition occurs such that the radiation slips a distance λ_r relative to the electrons after one undulator period. Thus, under certain favorable conditions, the interaction between the electron beam and the EM wave can be sustained and a net transfer of energy from electron beam to photon beam occurs. At some distance along the undulator, the radiation generated by the electron beam

First Hard X-ray FEL Facility

LCLS @ SLAC National Accelerator Laboratory

Claudio Pellegrini



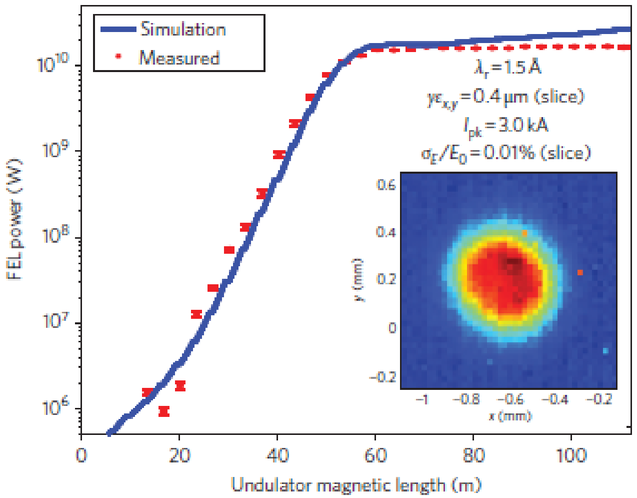
Paul Emma (right)

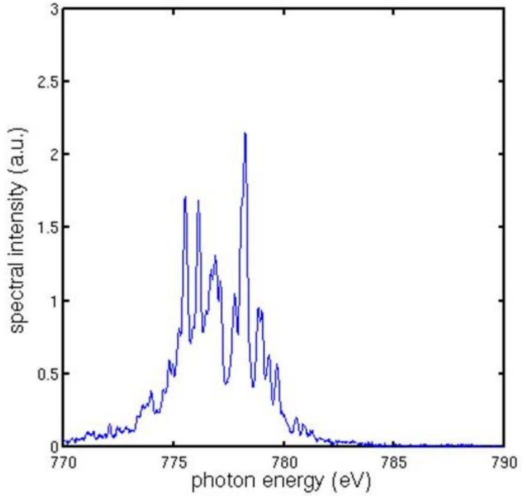
nature photonics | ARTICLES
PUBLISHED ONLINE: 1 AUGUST 2010 | DOI: 10.1038/NPHOTON.2010.176

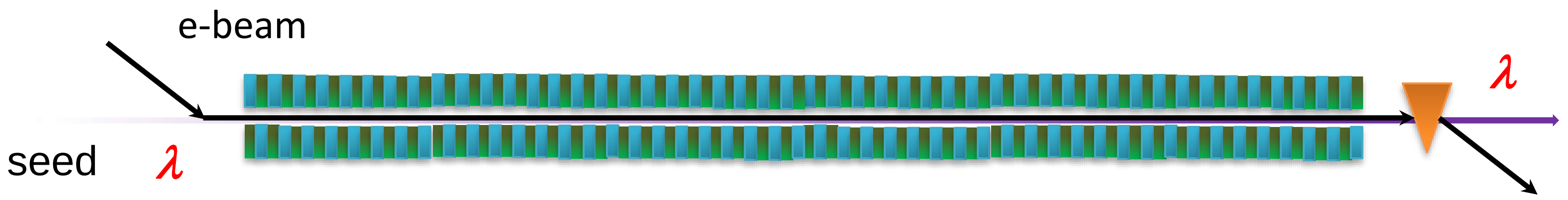
First lasing and operation of an ångstrom-wavelength free-electron laser

P. Emma[1]*, R. Akre[1], J. Arthur[1], R. Bionta[2], C. Bostedt[1], J. Bozek[1], A. Brachmann[1], P. Bucksbaum[1], R. Coffee[1], F.-J. Decker[1], Y. Ding[1], D. Dowell[1], S. Edstrom[1], A. Fisher[1], J. Frisch[1], S. Gilevich[1], J. Hastings[1], G. Hays[1], Ph. Hering[1], Z. Huang[1], R. Iverson[1], H. Loos[1], M. Messerschmidt[1], A. Miahnahri[1], S. Moeller[1], H.-D. Nuhn[1], G. Pile[3], D. Ratner[1], J. Rzepiela[1], D. Schultz[1], T. Smith[1], P. Stefan[1], H. Tompkins[1], J. Turner[1], J. Welch[1], W. White[1], J. Wu[1], G. Yocky[1] and J. Galayda[1]

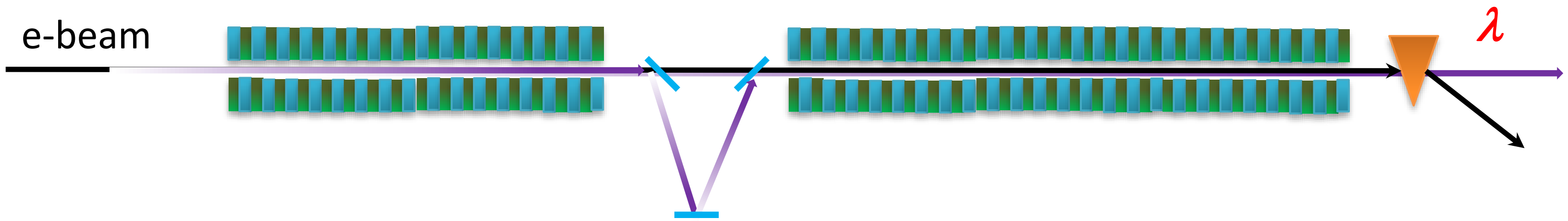
The recently commissioned Linac Coherent Light Source is an X-ray free-electron laser at the SLAC National Accelerator Laboratory. It produces coherent soft and hard X-rays with peak brightness nearly ten orders of magnitude beyond conventional synchrotron sources and a range of pulse durations from 500 to <10 fs (10^{-15} s). With these beam characteristics this light source is capable of imaging the structure and dynamics of matter at atomic size and timescales. The facility is now operating at X-ray wavelengths from 22 to 1.2 Å and is presently delivering this high-brilliance beam to a growing array of scientific researchers. We describe the operation and performance of this new 'fourth-generation light source'.





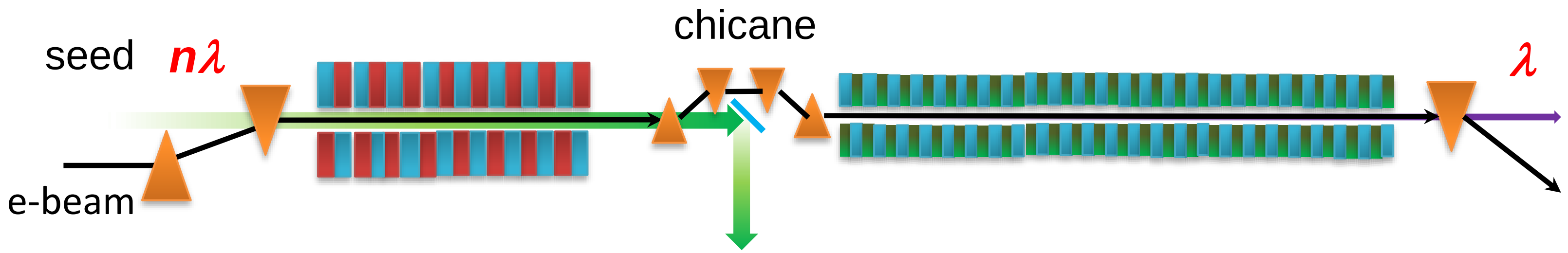


Direct seeding



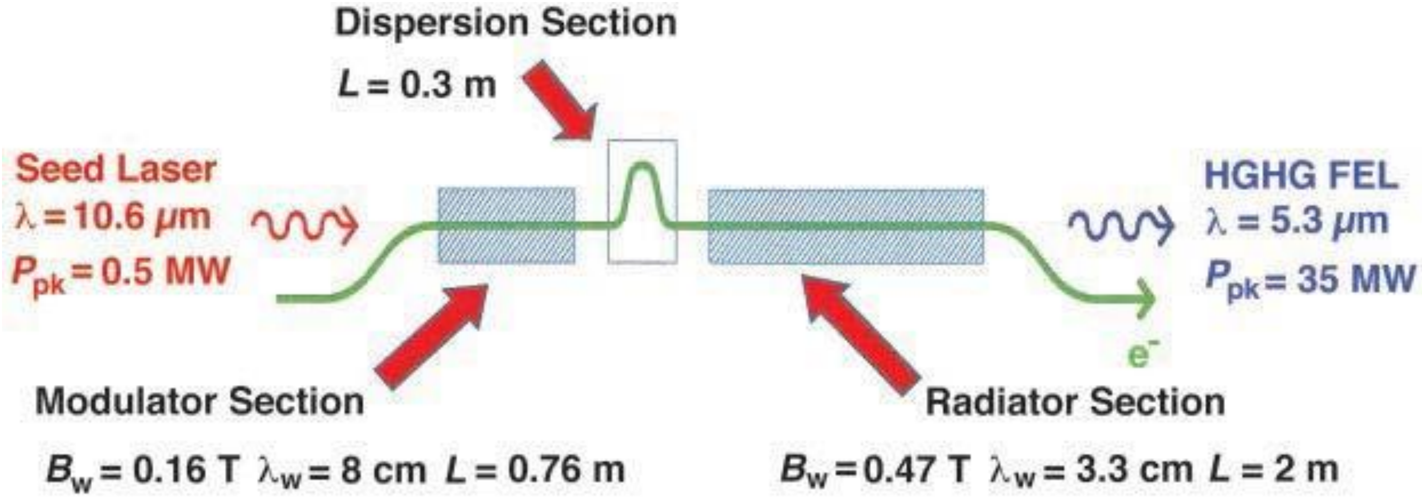
SASE with self-seeding

Spectral bandwidth can be reduced significantly. However, large fluctuation in output intensity is expected

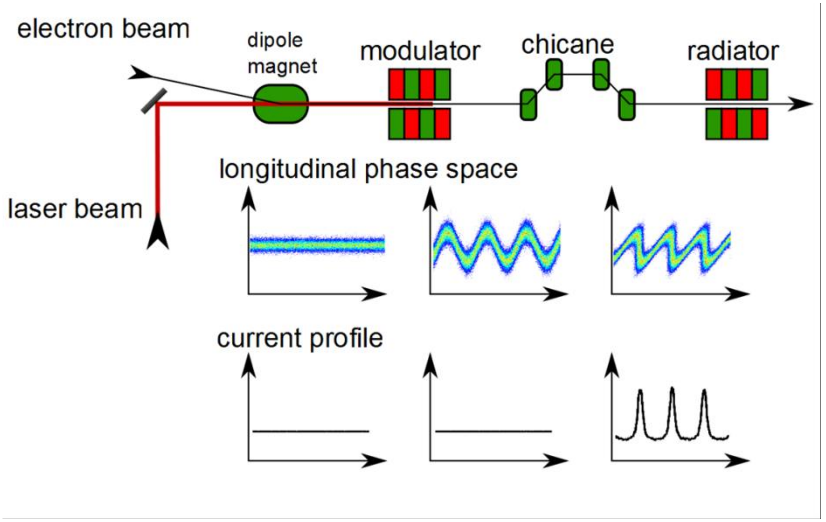


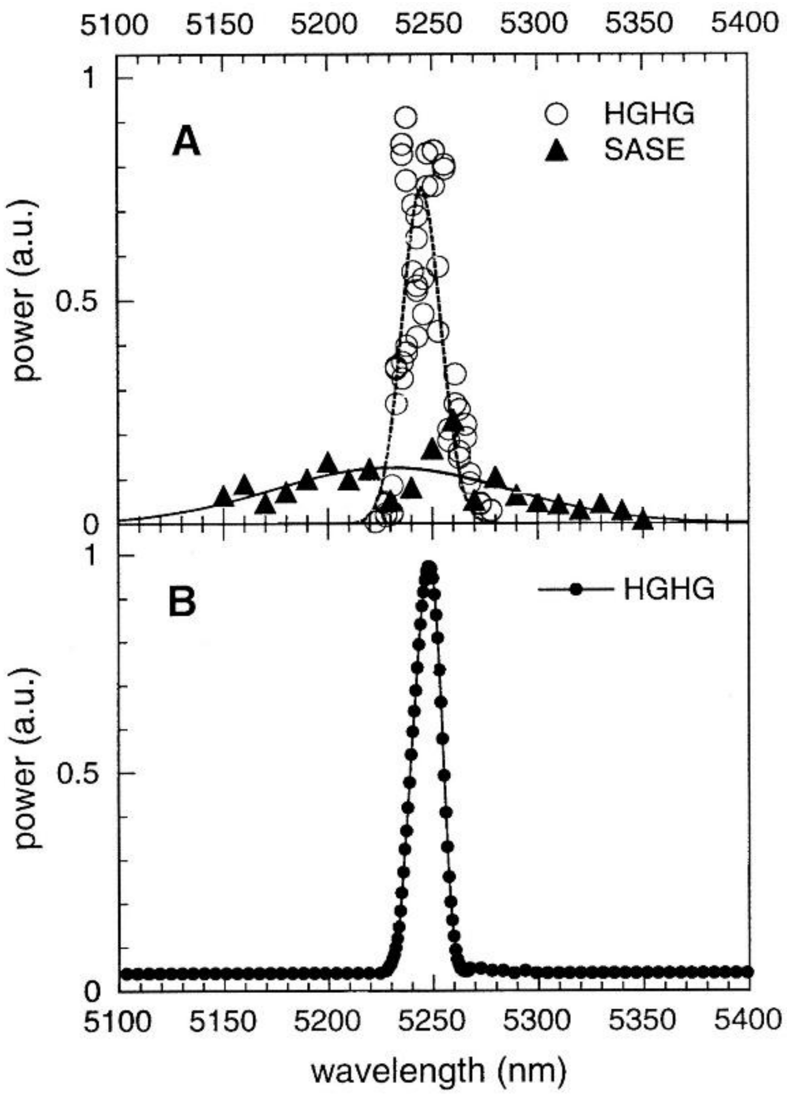
High Gain Harmonic Generation

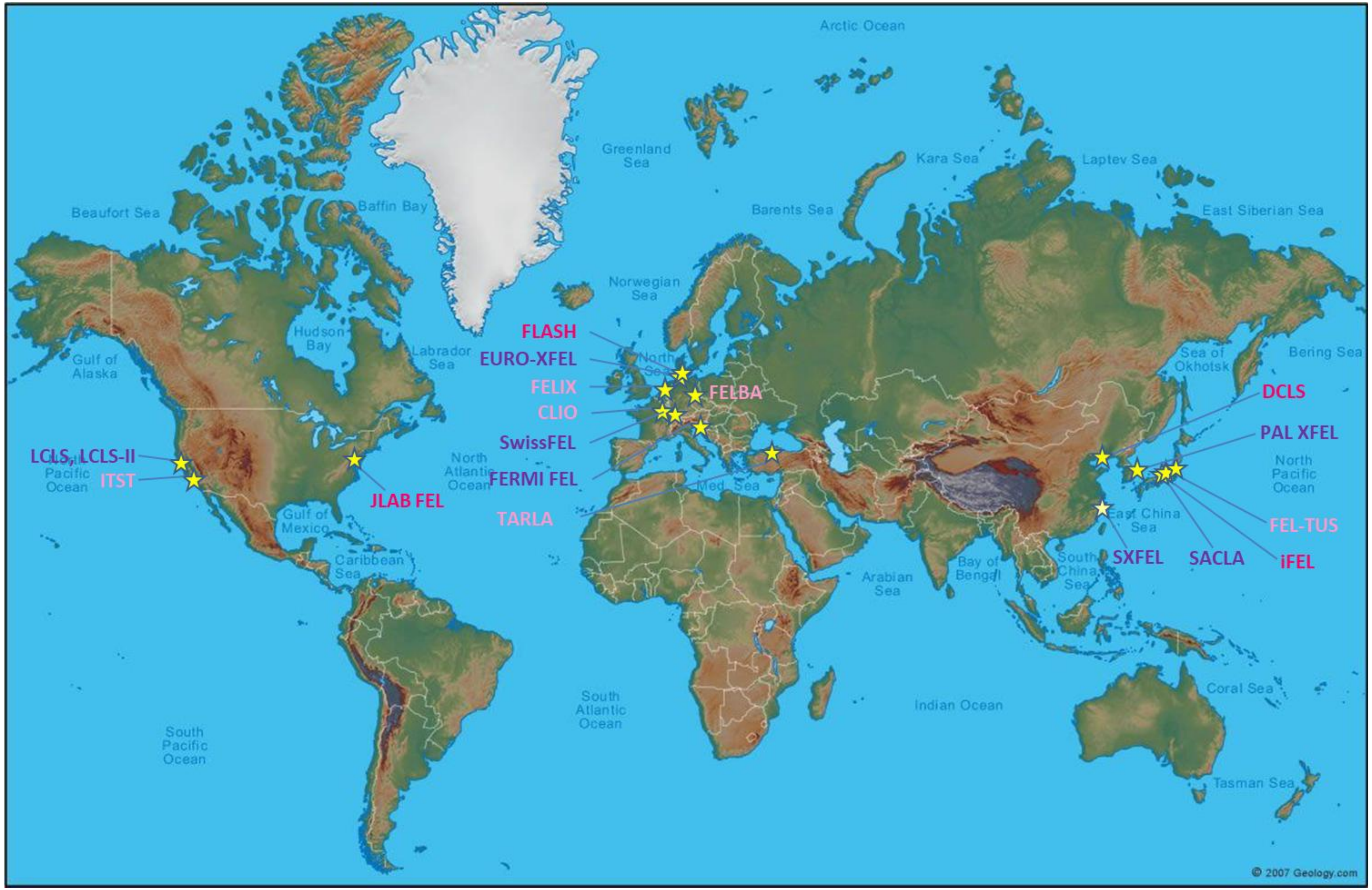
First HGHG Expt. @ Brookhaven National Lab



Li Hua Yu







FEL User Facilities for Scientific Research

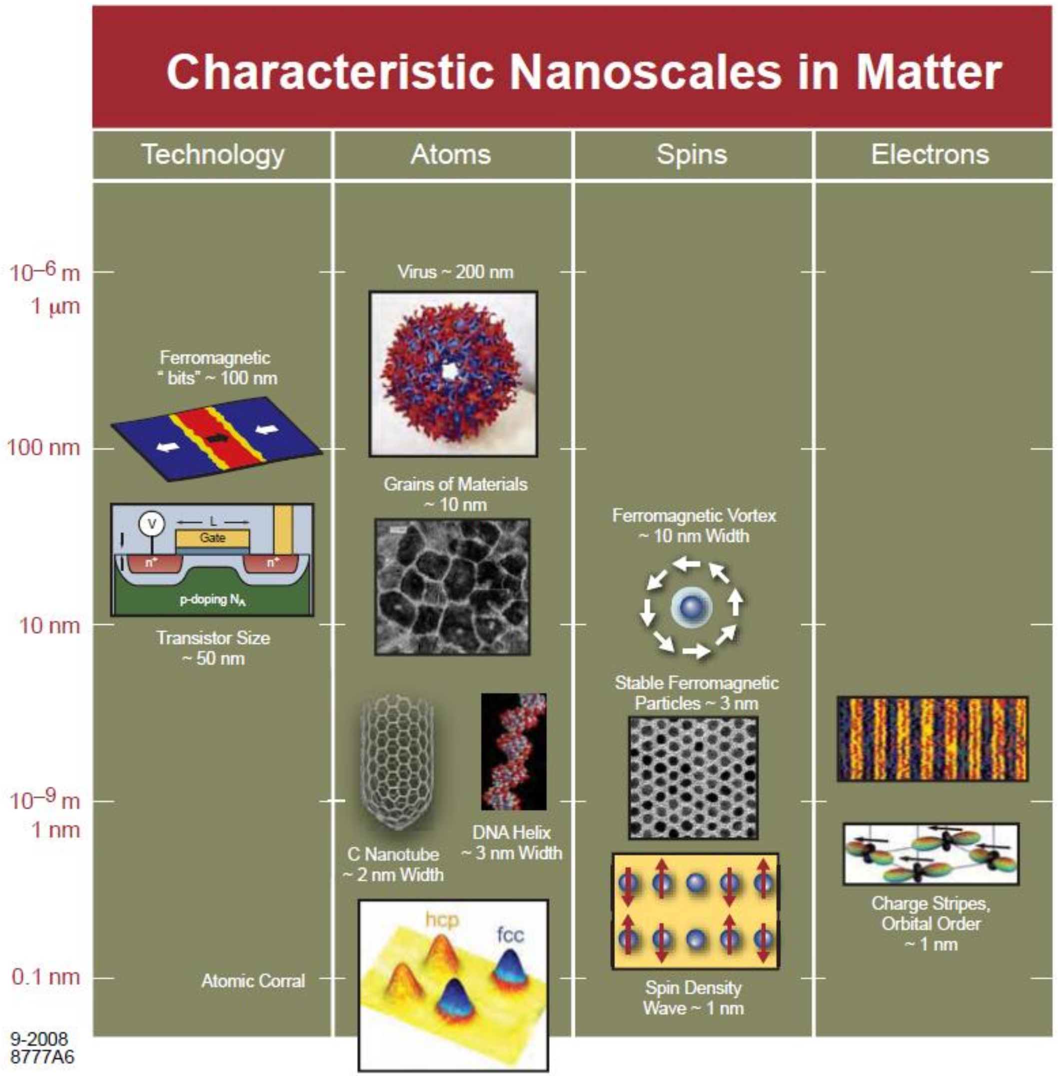
(Sources: http://sbfel3.ucsb.edu/www/vl_fel.html ; http://www.lightsouces.org)

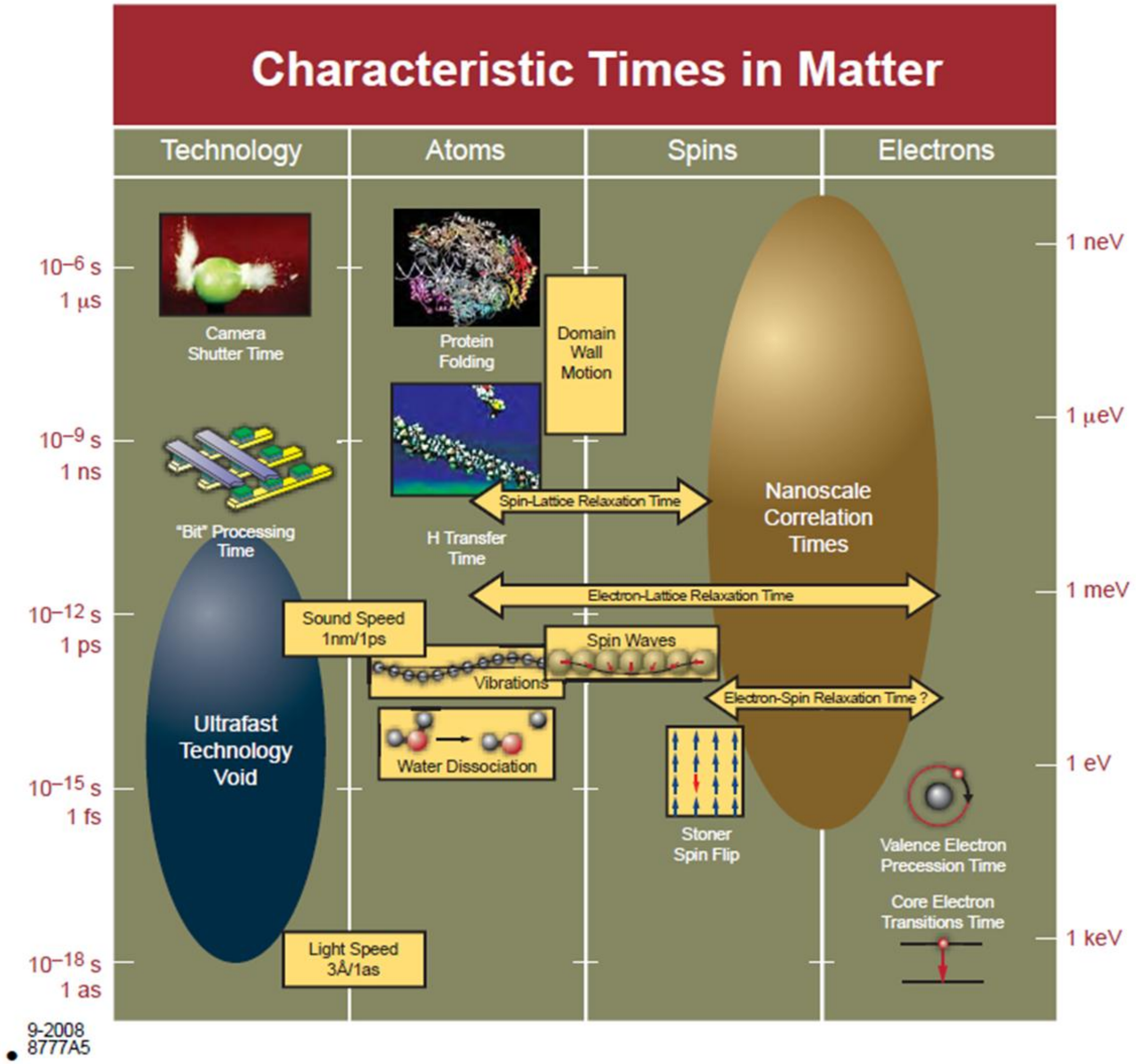
Free Electron Laser Facilities Worldwide

LOCATION	FACILITY NAME	WAVELENGTHS	Acc. Type
RIKEN, JAPAN	**SACLA**	**0.63 – 3 Å**	**NC Linac**
SLAC, USA	**LCLS**	**1.2 – 15 Å**	**NC Linac**
DESY, Germany	**FLASH**	**4.1 – 45 nm**	**SC Linac**
ELETTRA, Italy	**FERMI**	**4 – 100 nm**	**NC Linac**
Osaka U., Japan	iFEL	230 nm – 100 μm	NC Linac
Radboud U. Netherlands	FELIX	25 – 420 μm	NC Linac
LURE-Orsay, France	CLIO	3 – 150 μm	NC Linac
Jefferson Lab., USA	Jlab FEL	363 – 438 nm 3.2 – 4.8 μm	SC Linac
SUT, Japan	FEL-SUT	5 – 16 μm	NC Linac
FZ Rossendorf, Germany	FELBE	4 – 250 μm	SC Linac
UCSB, USA	ITST	30 μm – 2.5 mm	electrostatic
PSI, Switzerland	**Swiss FEL**	**1 – 70 Å**	**NC Linac**
DESY, Germany	**Euro-XFEL**	**0.5 – 47 Å**	**SC Linac**
Shanghai, China	**SXFEL**	**1.2 – 10 nm**	**NC Linac**
Dalian, China	**DCLS**	**50 – 150 nm**	**NC Linac**

high gain

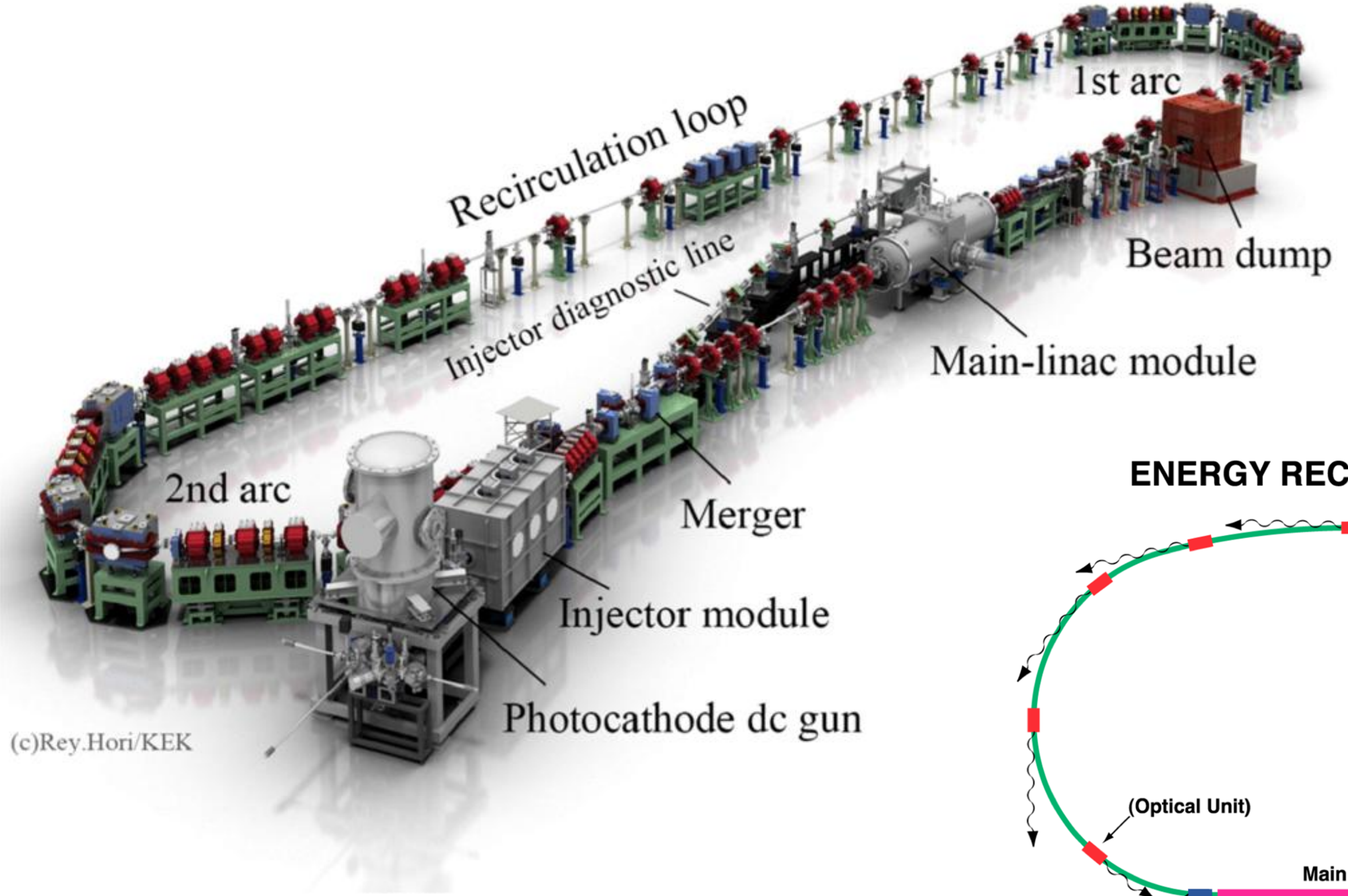
high gain


Characteristic Nanoscales in Matter
Technology
Atoms
Spins
Electrons
10−6 m
1 µm
100 nm
10 nm
10−9 m
1 nm
0.1 nm
Ferromagnetic " bits" ~ 100 nm
Gate
p-doping N_A
Transistor Size ~ 50 nm
Atomic Corral
Virus ~ 200 nm
Grains of Materials ~ 10 nm
C Nanotube ~ 2 nm Width
DNA Helix ~ 3 nm Width
hcp
fcc
Ferromagnetic Vortex ~ 10 nm Width
Stable Ferromagnetic Particles ~ 3 nm
Spin Density Wave ~ 1 nm
Charge Stripes, Orbital Order ~ 1 nm
9-2008
8777A6

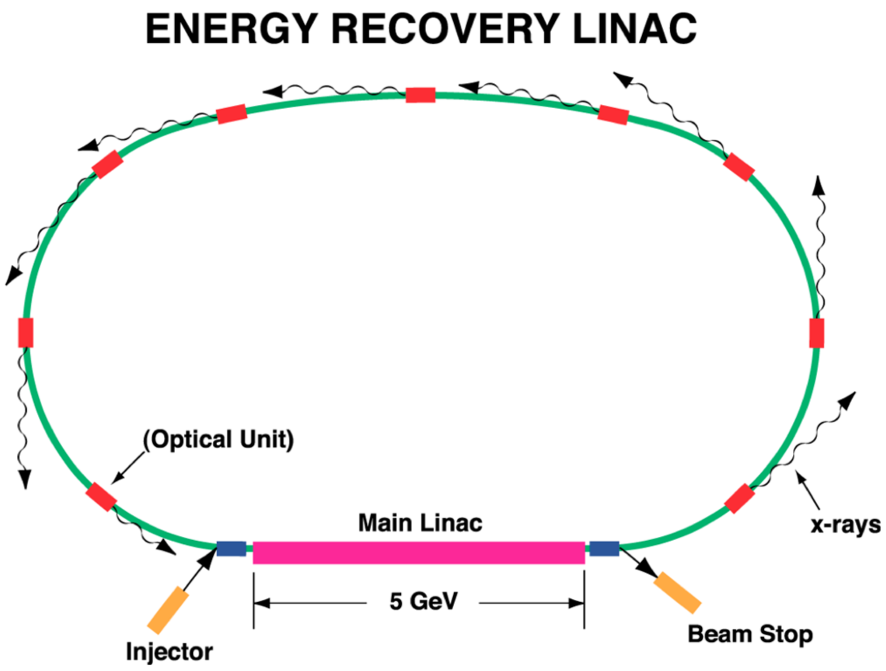

Characteristic Times in Matter
Technology
Atoms
Spins
Electrons
10−6 s
1 µs
10−9 s
1 ns
10−12 s
1 ps
10−15 s
1 fs
10−18 s
1 as
1 neV
1 µeV
1 meV
1 eV
1 keV
Camera Shutter Time
"Bit" Processing Time
Ultrafast Technology Void
Sound Speed 1nm/1ps
Light Speed 3Å/1as
Protein Folding
H Transfer Time
Vibrations
Water Dissociation
Domain Wall Motion
Spin-Lattice Relaxation Time
Electron-Lattice Relaxation Time
Spin Waves
Electron-Spin Relaxation Time ?
Stoner Spin Flip
Nanoscale Correlation Times
Valence Electron Precession Time
Core Electron Transitions Time
9-2008
8777A5

FEL在極紫外光微影製程的應用

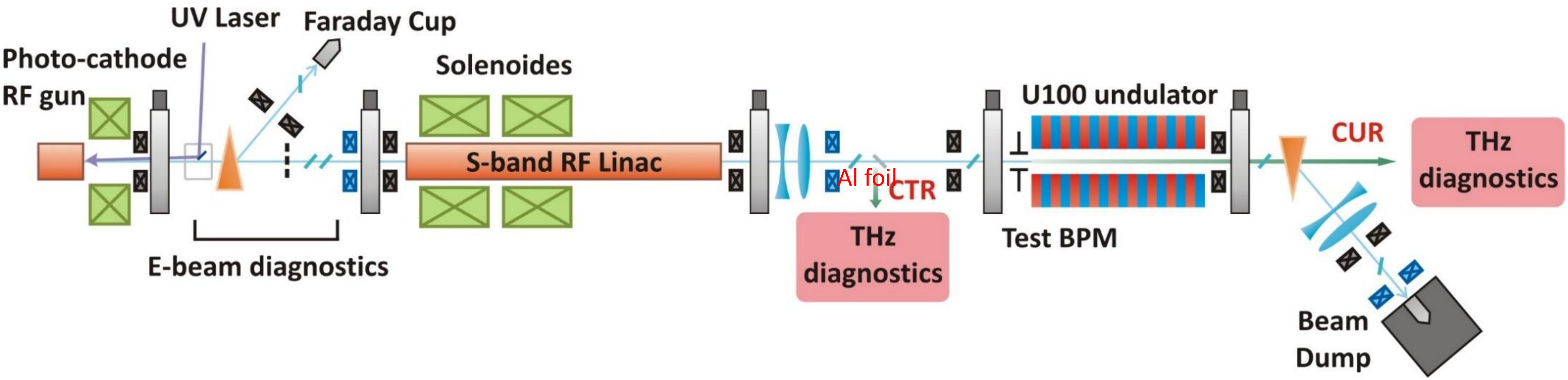
- 近年來，產業界在先進半導體的大批量量產技術上取得了顯著進步；主要原因之一是導入了穩定可靠的極紫外光微影製程（EUV lithography；EUVL）技術。
- 目前商用的EUVL微影製程系統以13.5奈米的雷射驅動電漿源（laser-produced plasma source；LPP）作為光源。他們設計的LPP利用二氧化碳雷射照射錫液滴（tin droplets），產生電漿並輻射出重複率為50仟赫、平均功率高達250瓦以上的極紫外光脈衝，供應系統後端的光學掃描子系統使用。
- 現今EUVL製程進行電路圖案轉印（patterning）的速率已達每小時超過200片晶圓。這在很大程度上滿足了晶片量產的需求。若要製程中增加圖案轉印的速率，增加EUV光源的平均功率是重要的方向。
- 然而，在LPP的發光過程中產生的殘留物也造成極紫外光收集鏡的污染而影響到鏡片的壽命。經過不斷改進保護收集鏡鍍膜的方式後，鏡面反射率的劣化速率（degradation rate）可以低於每十億發極紫外光脈衝0.1 %，尚有改進空間。



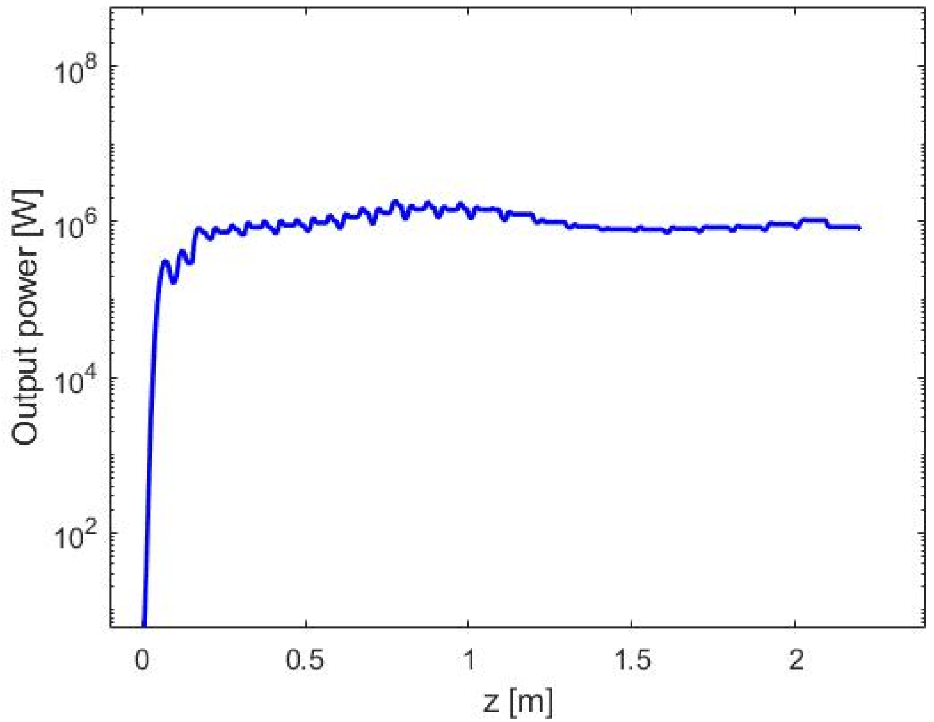
H. Sakai et al., Phys. Rev. Accel. Beams 28, 091603 (2025)



Superradiant THz Free Electron Laser @ NSRRC

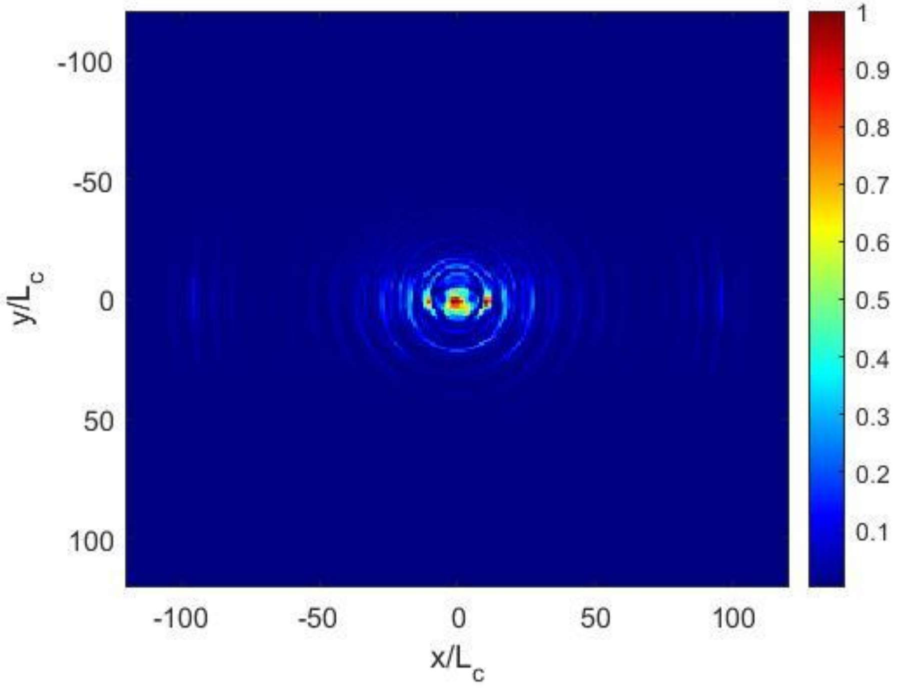


Simulation of Superradiant THz FEL



The radiation is immediately established in the U100 undulator and quickly achieves to its maximum value as we expected for super-radiant emission from an ultrashort drive. In this simulation, the parameter K of the U100 undulator is set at 4.6 and the radiation frequency is 2.6 THz.

- Since the bunch is short w.r.t. radiation length, simulation is done by unaveraged 3D FEL code – **PUFFIN**
- For start-to-end simulation, we use the RF compressed beam simulated by IMPACT-T as input for PUFFIN simulation



Spatial distribution of the THz radiation (c.f. L_c = 0.556 mm)

IR and THz FEL Facilities around the World

Facility Name	Location	Wavelength range	Type	Accelerator
CLIO	LURE-Orsay, France	3 – 120 μm	Oscillator	NC Linac
FELBE / TELBE	FZ Rossendorf, Germany	4 – 250 μm / 100 – 3000 μm	Oscillator / Superradiant	SC Linac
FHI FEL	Fritz Haber Institute, Germany	3 – 60 μm	Oscillator	NC Linac
FLASH THz Beamline	DESY, Germany	10 – 300 μm	Superradiant	SC Linac
PITZ THz SASE FEL	DESY, Germany	10 – 3000 μm	SASE	SC Linac
SABINA THz/IR FEL	SPARC Laboratory, Italy	10 – 100 μm	SASE	NC Linac
FELIX	Radboud U. Netherlands	3 – 1500 μm	Oscillator	NC Linac
TARLA	Gölbasi, Turky	2.5 – 250 μm	Oscillator	SC Linac
ALICE	Daresbury Lab., UK	4 –16 μm	Oscillator	SC Linac
FELiChEM	NSRL Hefei, China	2 – 200 μm	Oscillator	NC Linac
CAEP THz FEL	CAEP, Mianyang, China	71.4 – 447 μm / 686, 1344 μm	Oscillator /Superradiant	SC Linac
FEL-SUT	SUT, Japan	5 – 16 μm	Oscillator	NC Linac
FEL-TUS	Tokyo University of Science, Japan	5 – 1000 μm	Oscillator	NC Linac
iFEL	Osaka U., Japan	0.23 – 100 μm	Oscillator	NC Linac
KU FEL / THz CUR	Kyoto University, Japan	3.4 – 26 μm / 500 – 1873 μm	Oscillator / Superradiant	NC Linac
LEBRA	Nihon University, Japan	1– 6 μm	Oscillator	NC Linac
t-ACTS	Tohoku U., Japan	180 – 360 μm	Superradiant	NC Linac
NovoFEL	BINP Novosibirsk, Russia	8 – 340 μm	Oscillator	NC ERL
NSRRC THz FEL	NSRRC, Taiwan	214 – 500 μm	Superradiant	NC Linac
PCELL MIR FEL / THz FEL	CMU, Thailand	9.5 – 16.6 μm / 100 – 300 μm	Oscillator / Superradiant	NC Linac
ITST (UCSB FEL)	UCSB, USA	30 – 2500 μm	Oscillator	Electrostatic
Jlab FEL	Jefferson Laboratory, USA	1.5 – 14 μm	Ocillator	SC ERL

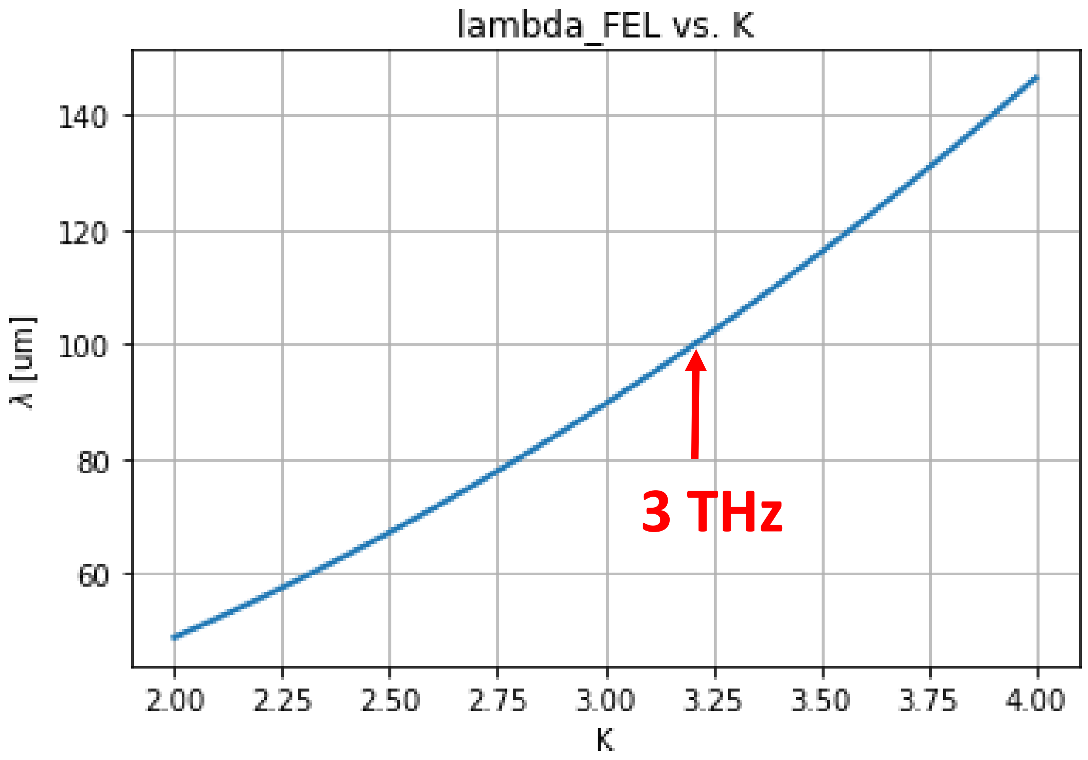
THz SASE FEL – an Example

Beam Parameters:

- Bunch charge: 2 nC
- Bunch duration: 13.33 ps
- Peak current: 150 A
- Beam energy: 15 MeV
- Normalized emittance: 3.0 mm-mrad
- Beam size: ~1 mm
- Energy spread: 100 keV

Undulator parameters:

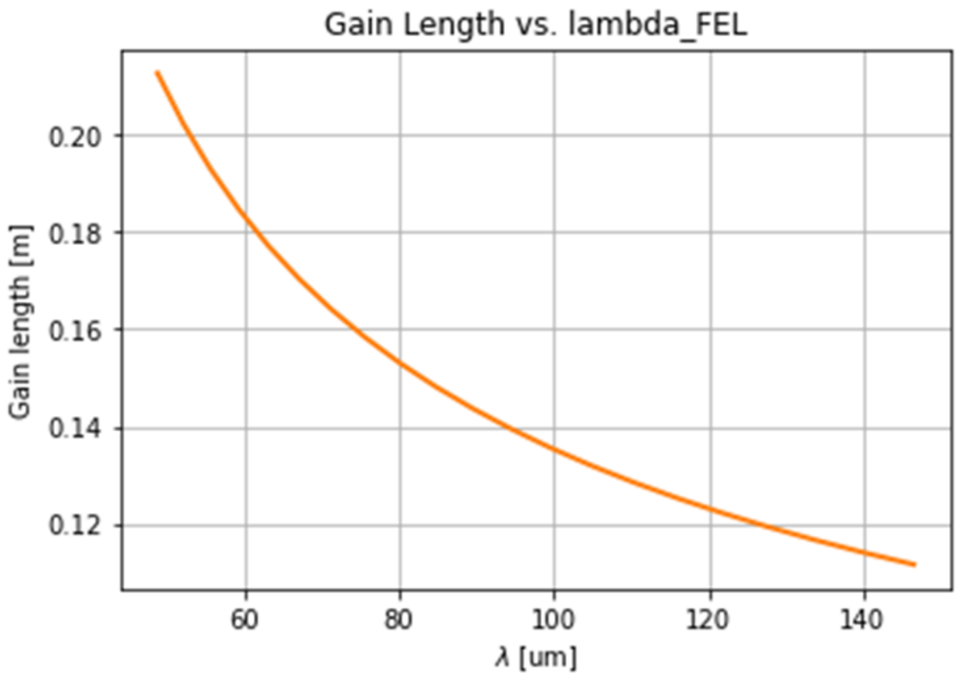
- Period length: 30 mm
- Number of periods: 100
- Undulator parameter: 2 – 4

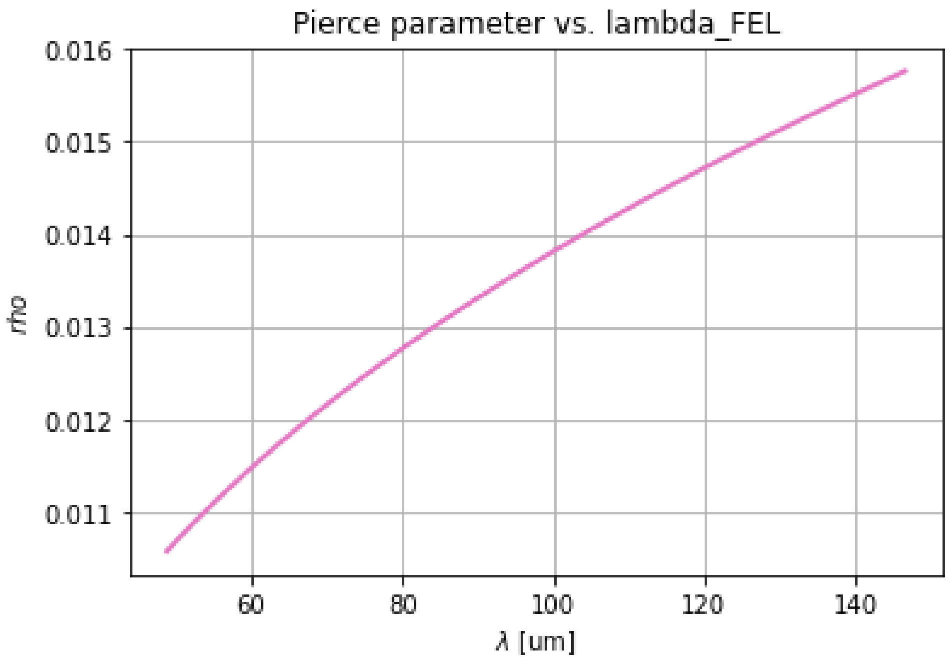


High bunch charge !!

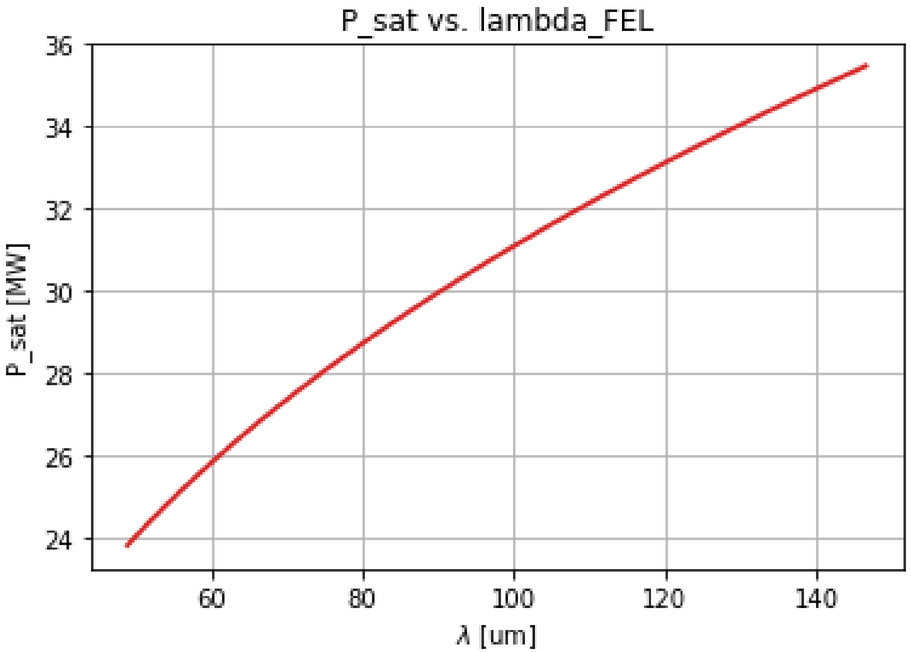
\# To achieve high peak power, the bunch should be long enough to contain as many micro-bunches as possible. Hence, it requires longer bunch at lower THz frequencies.

\# High peak current is required to make gain length short enough so that the FEL can reach saturation under fixed undulator length.

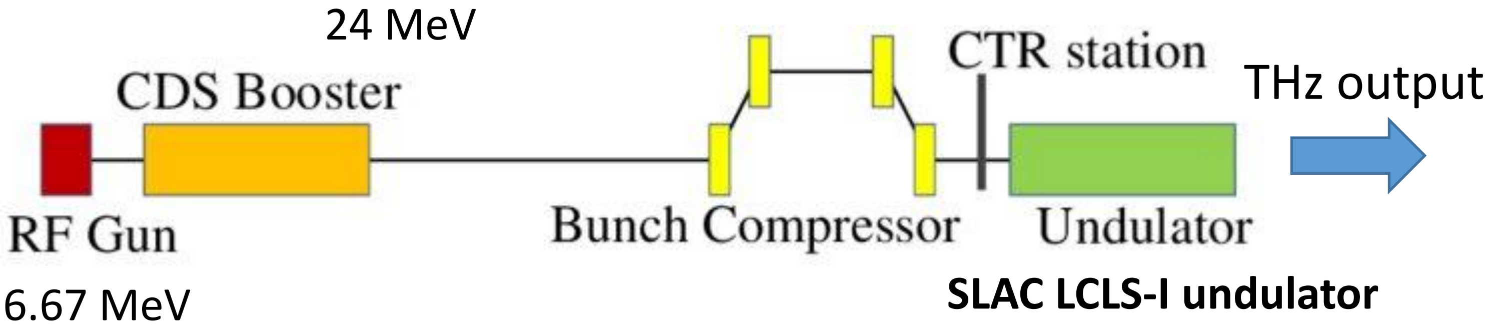


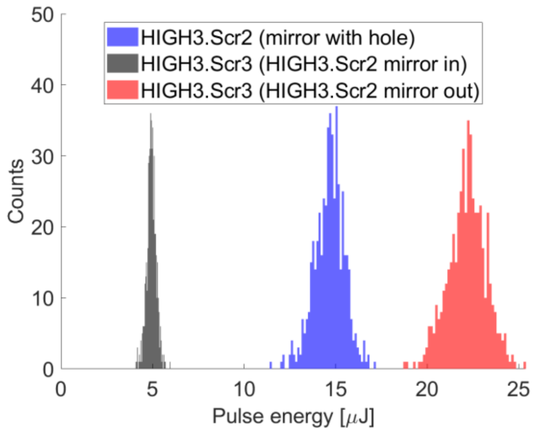


- Gain length, $L_g \sim 0.135$-m @ 100 μm wavelength
- $L_{sat} \sim 20\ L_g$ for SASE, should saturate in a 3-m long undulator
- FEL parameter, 0.0138 @ 100 m wavelength means that the interaction efficiency is about 1.38%
- Expected output power is 31 MW for 2.25 GW beam power.
- Corresponding pulse energy is 418 μJ !



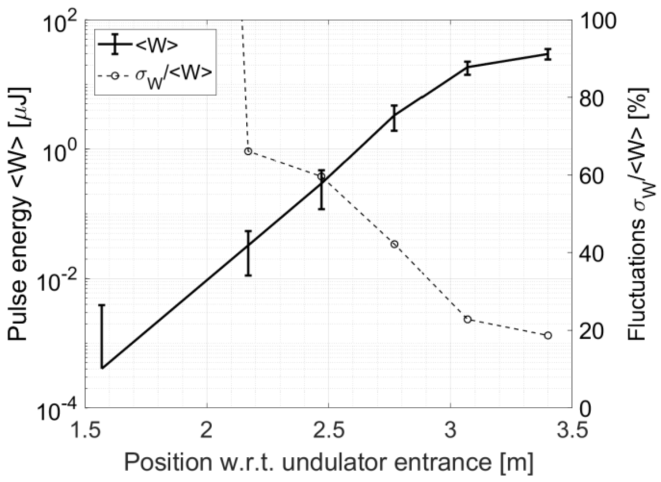
PITZ THz SASE FEL





- The THz SASE FEL prototype is under development at PITZ
- To be used for pump-probe experiment at European X-ray FEL
- 1.3 GHz photoinjector @ 60 MV/m accelerating gradient
- RF pulse duration of 1 ms at 10 Hz repetition rate
- 3.4-m long LCLS-I undulator (λu = 30 mm, 113 periods)

Parameter	Value
Bunch charge	3 nC
RMS bunch length	5.8±0.3 ps
Peak current	~165 A
Mean momentum	17 MeV/c
Projected momentum spread	98 keV/c
Maximum FEL pulse energy	29.67±5.54 μJ
Central wavelength	100 μm
Spectral bandwidth	≤12 μm FWHM



Boonpornprasert, P. et al., in IPAC2023, Venice, 2023.